Unit 3

Define strongly connected graph.
Answer:
A digraph is said to be strongly connected graph, if there is a path between every pair of vertices in the digraph.

Define a regular graph. Can a complete graph be a regular graph?
Ans: A graph is said to be regular if all the vertices are of same degree.
Yes a complete graph is always a regular graph.

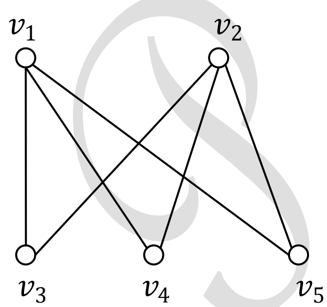
State the handshaking theorem.
Solution:
If $G(V, E)$ is an undirected graph with e edges, then

$$\sum_i \deg v_i = 2e.$$

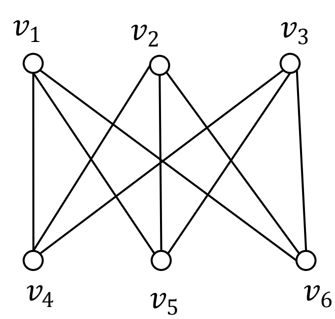
State the necessary and sufficient conditions for the existence of an Eulerian path in a connected graph.
Answer:
A connected graph has an Euler path but not Euler circuit if and only if it has exactly two vertices of odd degree.

Define Pseudo graph.
Answer:
A graph with self loops and parallel edges is called Pseudo graphs.

Draw a complete bipartite graph of $K_{2,3}$ and $K_{3,3}$.
Solution:



$K_{2,3}$

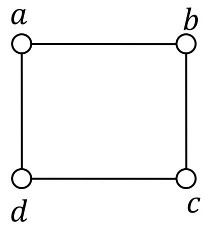


$K_{3,3}$

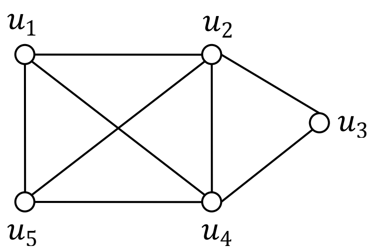
Define isomorphism of two graphs.
Ans:
Two graphs $G_1$and G_2 are said to be isomorphic to each other, if there exists one to one correspondence between the vertex sets preserves adjacency of the vertices.

Give an example of an Euler graph.

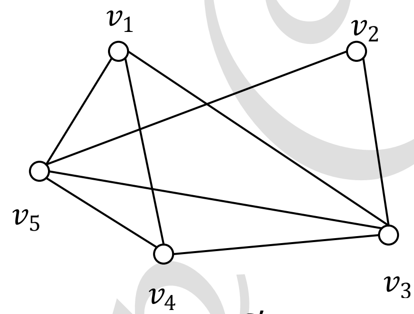
Ans:



Examine whether the following pair of graphs are isomorphic. If not isomorphic, give the reasons.



G



G'

Solution:
In G, the number of vertices is 5, the number of edges is 8.

$$\deg(u_1) = 3, \deg(u_2) = 4, \deg(u_3) = 2, \deg(u_4) = 4, \deg(u_5) = 3$$

In G', the number of vertices is 5, the number of edges is 8.

$$\deg(v_1) = 3, \deg(v_2) = 2, \deg(v_3) = 4, \deg(v_4) = 3, \deg(v_5) = 4$$

There are same number of vertices and edges in both the graph G and G'.
Here in both graphs G and G', two vertices are of degree 3, two vertices are of degree 4, and one vertex is of degree 2.

$$u_1 \to v_1, u_2 \to v_5, u_3 \to v_2, u_4 \to v_3, u_5 \to v_4$$

There is one to one correspondences between the graphs G and G'.
∴ The graphs $G$and G' are isomorphic.

Let G be a simple undirected graph with n vertices. Let u and v be two non adjacent vertices in G such that $\deg(u) + \deg(v) \geq n$ in . Show that G is Hamiltonian if and only if $G + uv$ is Hamiltonian.
Solution:
If G is Hamiltonian, then obviously $G + uv$ is Hamiltonian.
Conversely, suppose that $G + uv$ is Hamiltonian, but G is not.
[**Dirac theorem :** If G is a simple graph with at least three vertices and $\delta(G) \geq \frac{|V(G)|}{2}$ then G is Hamiltonian.]
Then by Dirac theorem, we have

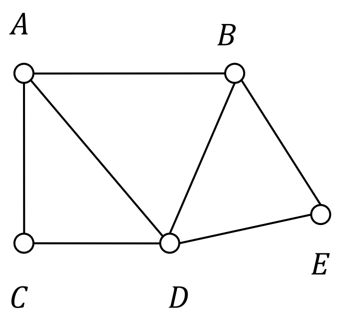
$$\deg(u) < \frac{n}{2} \ and \deg(v) < \frac{n}{2}$$

$$\therefore \deg(u) + \deg(v) < \frac{n}{2} + \frac{n}{2}$$

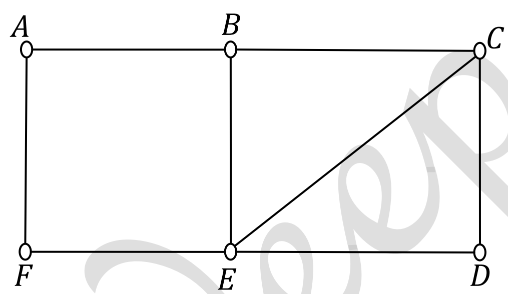
$$\therefore \deg(u) + \deg(v) < n$$

which is a contradiction to our assumption $G$is not Hamiltonian.
$\therefore$ G is Hamiltonian.
Thus $G+uv$ is Hamiltonian implies G is Hamiltonian.

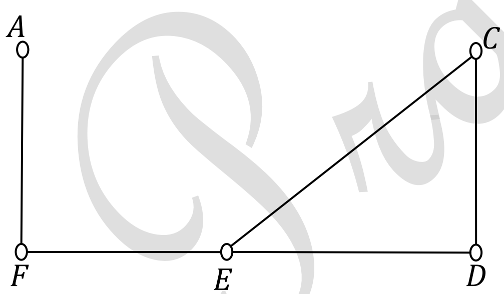
Draw the graph with 5 vertices, A, B, C, D, E such that $deg(A) = 3$, B is an odd vertex, $deg(C) = 2$ and $D$and E are adjacent.

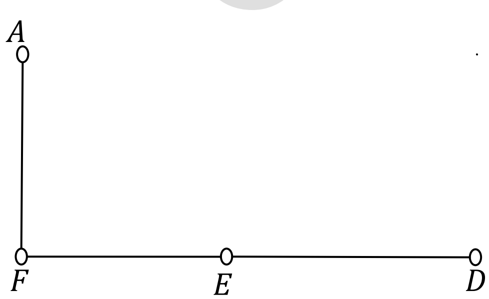


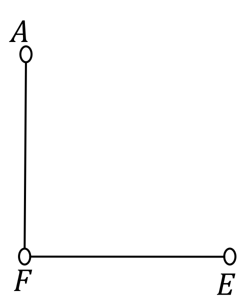
Find the all the connected sub graph obtained form the graph given in the following Figure, by deleting each vertex. List out the simple paths from A to in each sub graph.



Solution:
The connected sub graph obtained form the graph given in the Figure, by deleting each vertex are





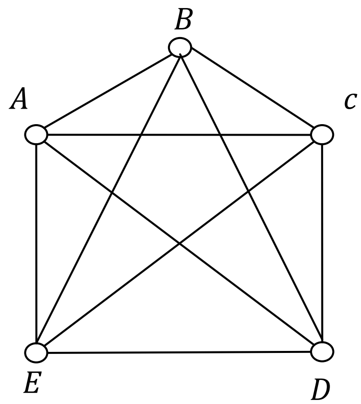


The simple paths from $\boldsymbol{A}$ to in each sub graph is
(1) $A \to F \to E \to D \to C,\ A \to F \to E \to C \to D$
(2) $A \to F \to E \to D$
(3) $A \to F \to E$
(4) $A \to F$

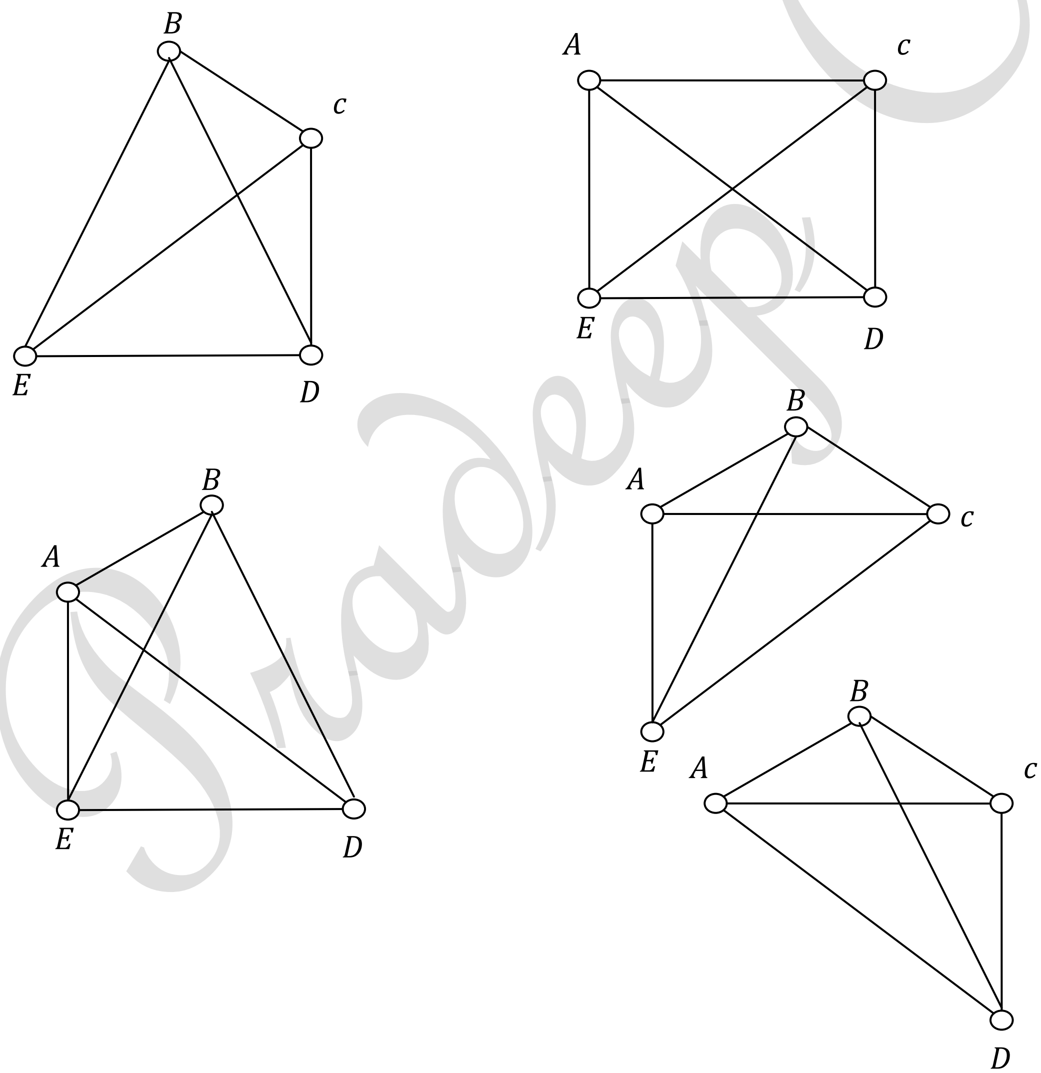
Prove that a connected graph G is Eulerian if and only if all the vertices are on even degree.
Proof:
Suppose, G is an Euler graph. G contains an Eulerian circuit. While traversing through the circuit a vertex v is incident by two edges with one we entered and other exited. This is true, for all the vertices, because it is a circuit. Thus the degree of every vertex is even.
Conversely, suppose that all vertices of G are of even degree, we have to prove that G is an Euler graph.
Construct a circuit starting at an arbitrary vertex v and going through the edge of G such that no edge id repeated. Because, each vertex is of even degree, we can exit from each end, every vertex we enter, the tracing can stop only at vertex v. Name the circuit as h. If h covers all edges of G, then G contains Euler circuit, and hence G is an Euler graph. If h does not cover all edges of G then remove all edges of h from G and obtain the remaining graph $G^{'}$. Since G and G' contains all the vertex of even degree. Every vertex in G' is also of even degree. Since G is connected, h will touch G' atleast one vertex $v^{'}$. Starting from v' we can again construct a new circuit h' in G'. This will terminate only at $v^{'}$, because every vertex in $G^{'}$ is of even degree. Now, this circuit h' combined with h forms a circuit starts and ends at v and has more edges than h, this process is repeated until we obtain a circuit covering all edges of G. Thus G is an Euler graph.

Show that graph $\boldsymbol{G}$ is disconnected if and only if its vertex set $\boldsymbol{V}$ can be partitioned into two nonempty subsets $\boldsymbol{V_1}$ and $\boldsymbol{V_2}$ such that there exists no edge in $\boldsymbol{G}$ whose one end vertex is in $\boldsymbol{V_1}$ and the other in $\boldsymbol{V_2}$.
Proof:
Suppose that such a partition exists. Consider two arbitrary vertices a and b of G such that $a \in V_1$ and $b \in V_2$. No path can exist between vertices a and b. Otherwise, there would be at least one edge whose one end vertex be in V_1 and the other in V_2. Hence if partition exists, G is not connected.
Conversely, let G be a disconnected graph.
Consider a vertex a in G. Let V_1 be the set of all vertices that are joined by paths to a. Since G is disconnected, V_1 does not include all vertices of G. The remaining vertices will form a set V_2. No vertex in V_1 is joined to any vertex in V_2 by an edge. Hence the partition.

Draw the complete graph $\boldsymbol{K_5}$ with vertices , $\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ and $\boldsymbol{E}$. Draw all complete sub graph of $\boldsymbol{K_5}$ with 4 vertices.
Solution:
A complete graph with five vertices K_5 is shown below



Complete sub graph of K_5 with 4 vertices are



If all the vertices of an undirected graph are each of degree k, show that the number of edges of the graph is a multiple of k .

Solution:

Let $G(V, E)$ be a graph with n vertices and e edges.

Let $v_1, v_2, \dots, v_n$ be the n vertices.

Given that all the vertices of G are each of degree k.

$$\deg(v_1) = \deg(v_2) = \deg(v_3) = \cdots = \deg(v_k) = k$$

By handshaking theorem,

$$\sum_{i=1}^{n} \deg(v_i) = 2e$$

$$\deg(v_1) + \deg(v_2) + \deg(v_3) + \cdots + \deg(v_n) = 2e$$

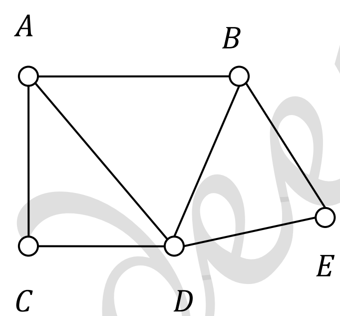
$$k + k + k + \cdots ntimes = 2e$$

$$nk = 2e$$

$$e = k\left(\frac{n}{2}\right)$$

∴ The number of edges of the graph G is a multiple of .

Draw the graph with 5 vertices, A, B, C, D, E such that $eg(A) = 3$, B is an odd vertex, $deg(C) = 2$ and $D$and E are adjacent.



The adjacency matrices of two pairs of graph as given below. Examine the isomorphism of *G* and *H* by finding a permutation matrix. $\mathbf{A_G} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \end{pmatrix}, \mathbf{A_H} = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \end{pmatrix}$

Solution:

We know that two simple graphs G_1 and G_2 are isomorphic iff their adjacency matrices A_1 and A_2 are related by

$$PA_1P^T = A_2$$

[A matrix whose rows are the rows of the unit matrix, but not necessarily in their natural order, is called Permutation matrix.]

$$\mathrm{A_G} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \mathrm{A_H} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

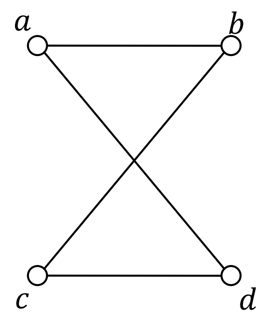
$$PA_GP^T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} A_H$$

$$PA_GP^T = A_H$$

∴ The two graphs G and H are isomorphic.

How many paths of length four are there from a to d in the simple graph G given below.



Solution: The adjacency matrix for the given graph is

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

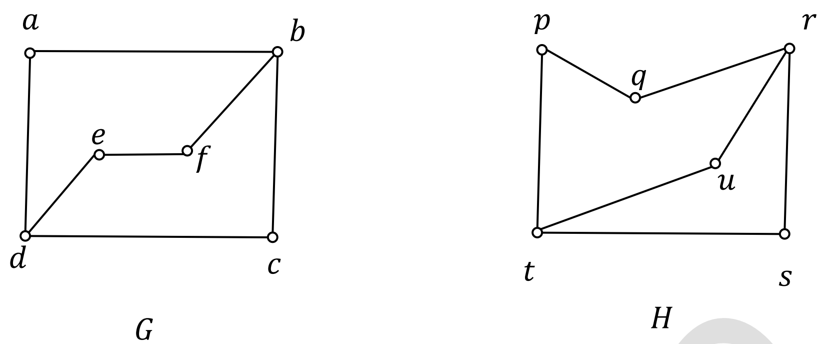
$$A^2 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix} \end{array}$$

$$A^4 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 8 & 0 & 8 & 0 \\ 0 & 8 & 0 & 8 \\ 8 & 0 & 8 & 0 \\ 0 & 8 & 0 & 8 \end{pmatrix} \end{array}$$

Since the value in the a^{th} row and d^{th} column in A^4 is 0.
∴ There is no path from a to d of length 4.

Show that the complete graph with n vertices K_n has a Hamiltonian circuit whenever $n \geq 3$.
Proof:
In a complete graph K_n, every vertex is adjacent to every other vertex in the graph.
Therefore a path can be formed starting from any vertex and traverse through all the vertices of K_n and reach the same vertex to form a circuit without traversing through any vertex more than once except the terminal vertex. This circuit is called the Hamiltonian circuit.
∴ The complete graph with n vertices K_n has a Hamiltonian circuit whenever $n \geq 3$.

Determine whether the graphs G and H given below are isomorphic.



Solution:

The two graphs G and H have same number of vertices and same number of edges.

$$\deg(a) = 2, \deg(b) = 3, \deg(c) = 2, \deg(d) = 3, \deg(e) = 2, \deg(f) = 2$$
$$\deg(p) = 2, \deg(q) = 2, \deg(r) = 3, \deg(s) = 2, \deg(t) = 3, \deg(u) = 2$$

Here in both the graphs two vertices are of degree three and remaining vertices are of degree two.

The mapping between two graphs are $a \to u, b \to r, c \to s, d \to t, e \to p$ and $f \to q$.

There is one to one correspondence between the adjacency of the vertices between the two graphs.

Therefore the two graphs are isomorphic.

Prove that an undirected graph has an even number of vertices of odd degree.

Proof:

By Handshaking theorem, we know that

If the graph G has n vertices and e edges then

$$\sum_{i=1}^{n} \deg(v_i) = 2e \Rightarrow \sum_{i=1}^{n} \deg(v_i) = even\ number$$
$$\sum_{odd} \deg(v_i) + \sum_{even} \deg(v_i) = even\ number$$
$$where \sum_{odd} \deg(v_i)\ \ is\ sum\ of\ vertices\ with\ odd\ degree$$
$$\sum_{even} \deg(v_i)\ \ is\ sum\ of\ vertices\ with\ even\ degree$$

$$\sum_{odd} \deg(v_i) + even\ number = even\ number$$

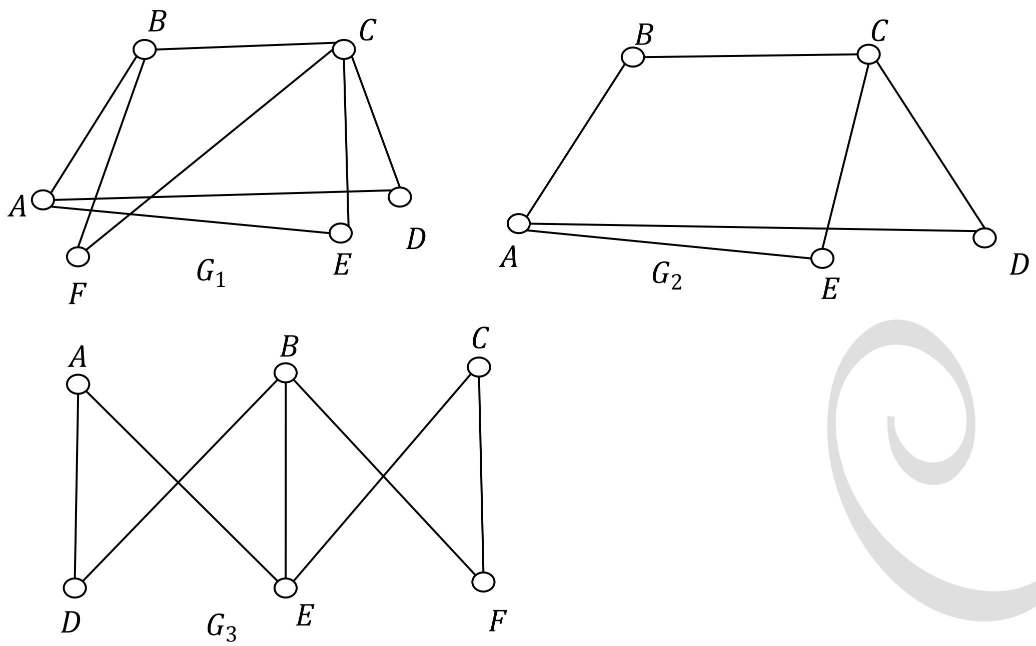
(Since sum of even numbers is an even number)

$$\sum_{odd} \deg(v_i) = even\ number$$

Sum of even number of odd numbers is an even number.

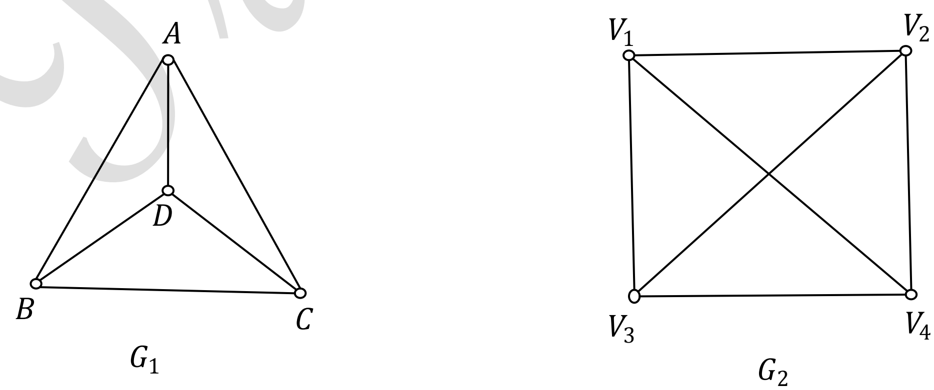
∴ An undirected graph has an even number of vertices of odd degree.

Determine which of the following graphs are bipartite and which are not. If a graph is bipartite, state if it is completely bipartite.



Solution:
In the graph G_1, Since there is no edges between D, E and F, let us take it as one vertex set $V_1 = \{D, E, F\}$. Obviously the other vertex set will be $V_2 = \{A, B, C\}$.
Since there are edges between A and B, and B and C.
$\therefore G_1$ is not a bipartite graph.
In the graph G_2, Let $V_1 = \{A, C\}$ and $V_2 = \{B, D, E\}$. Since there is no edge between the vertices in the same vertex set, G_2 is a bipartite graph. Since there are edges between every vertices in the vertex set V_1 to every vertices in the vertex set V_2, G_2 is completely bipartite.
In the graph G_3, Let $V_1 = \{A, B, C\}$ and $V_2 = \{D, E, F\}$. Since there is no edge between the vertices in the same vertex set, G_3 is a bipartite graph. Since there is no edge between A and F, C and D, where $A, C \in V_1$ and $D, F \in V_2$. $\therefore$ G_3 is not completely bipartite.

Using circuits, examine whether the following pairs of graphs G_1, G_2 given below are isomorphic or not:



Solution:
In G_1, the number of vertices is 4, the number of edges is 6.

$$\deg(A) = 3, \deg(B) = 3, \deg(C) = 3, \deg(D) = 3$$

In G_2, the number of vertices is 4, the number of edges is 6.

$$\deg(V_1) = 3, \deg(V_2) = 3, \deg(V_3) = 3, \deg(V_4) = 3$$

There are same number of vertices and edges in both the graph G_1 and G_2.
Here in both graphs G_1 and G_2, all vertices are of degree 3.
The mapping between the vertices of two graphs is given below

$$A \to V_1, B \to V_2, C \to V_3, D \to V_4$$

There is one to one correspondences between the adjacency of the vertices in the graphs G_1 and G_2.
∴ The graphs G_1 and G_2 are isomorphic.

Prove that the maximum number of edges in a simple disconnected graph with $\boldsymbol{n}$ vertices and $\boldsymbol{k}$ components is

$$\frac{(\boldsymbol{n}-\boldsymbol{k})(\boldsymbol{n}-\boldsymbol{k}+\boldsymbol{1})}{\boldsymbol{2}}.$$

Solution:
Let n_i be the number of vertices in i^{th} component.

$$\sum_{i=1}^{k} n_i = n \ldots (1)$$

$$\sum_{i=1}^{k}(n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k \ldots (2)\ [from\ (1)]$$

Squaring on both sides of (2), we get

$$\left(\sum_{i=1}^{k}(n_i - 1)\right)^2 = (n-k)^2$$

$$\sum_{i=1}^{k}(n_i - 1)^2 + \sum_{i \neq j}^{k}(n_i - 1)(n_j - 1) = (n-k)^2$$

$$\sum_{i=1}^{k}(n_i - 1)^2 \leq (n-k)^2$$

$$\sum_{i=1}^{k}(n_i^2 - 2n_i + 1) \leq n^2 - 2nk + k^2$$

$$\sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + \sum_{i=1}^{k} 1 \leq n^2 - 2nk + k^2$$

$$\sum_{i=1}^{k} n_i^2 - 2n + k \leq n^2 - 2nk + k^2 \qquad [from\ (1)]$$

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - 2nk + k^2 + 2n - k \ \ldots (3)$$

The maximum number of edges in i^{th} component is

$$\frac{n_i(n_i - 1)}{2}$$

∴The maximum number of edges in the graph is

$$\sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} = \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}\sum_{i=1}^{k} n_i$$

$$= \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}n \qquad [from\ (1)]$$

$$\leq \frac{1}{2}(n^2 - 2nk + k^2 + 2n - k) - \frac{1}{2}n \qquad [from\ (3)]$$

$$\leq \frac{1}{2}(n^2 - 2nk + k^2 + n - k)$$

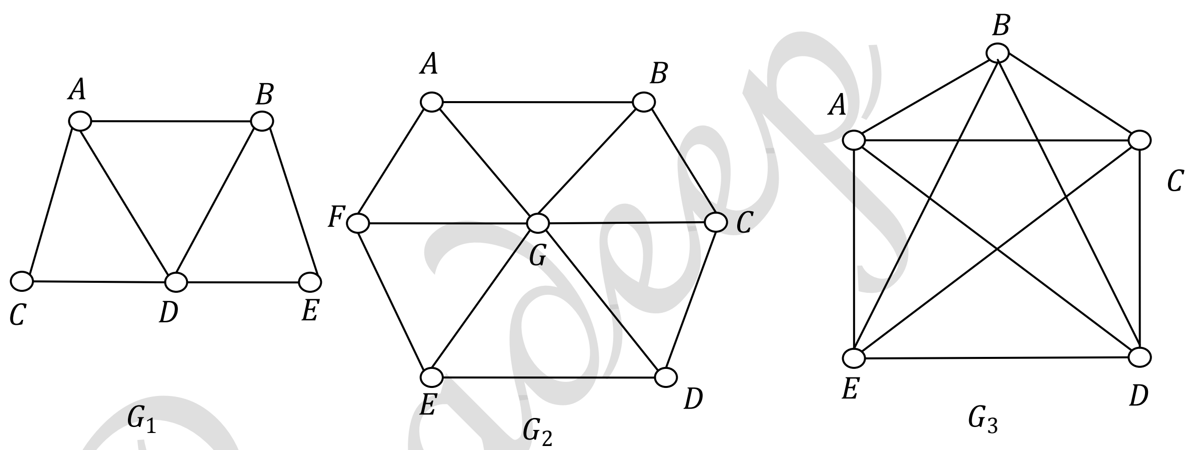
$$\leq \frac{1}{2}((n-k)^2 + n - k)$$

$$\leq \frac{1}{2}(n-k)(n-k+1)$$

∴The maximum number of edges in the graph is

$$\leq \frac{(n-k)(n-k+1)}{2}$$

Find an Euler path or an Euler circuit, if it exists in each of the three graphs below. If it does not exist, explain why?



Solution:
In graph G_1, $\deg(A) = 3, \deg(B) = 3, \deg(C) = 2, \deg(D) = 4, \deg(E) = 2$
The graph G_1 contains only two vertices of odd degree and all the other vertices are of even degree.
∴ G_1 has an Eulerian path but not Eulerian circuit.
The Eulerian path for graph G_1 is $A \to B \to E \to D \to C \to A \to D \to B$
In graph G_2, $\deg(A) = 3, \deg(B) = 3, \deg(C) = 3, \deg(D) = 3, \deg(E) = 3$, $\deg(F) = 3, \deg(G) = 6$
The graph G_2 contains only one vertex of even degree and all the other vertices are of odd degree.
∴ G_2 don’t contain neither Eulerian path nor Eulerian circuit.
In graph G_3, $\deg(A) = 4, \deg(B) = 4, \deg(C) = 4, \deg(D) = 4, \deg(E) = 4$
The graph G_3 has all the vertices are of even degree.
∴ G_3 has an Eulerian circuit.
The Eulerian circuit for graph G_3 is $A \to B \to C \to D \to E \to A \to D \to B \to E \to C \to A$