Unit 2 Combinatorics

State pigeonhole principle.

Solution:

If k pigeons are assigned to n pigeonholes and $n < k$ then there is at least one pigeonhole containing more than one pigeons.

Find the recurrence relation satisfying the equation $y_n = A(3)^n + B(-4)^n$.

Solution: $y_n = A(3)^n + B(-4)^n$.

$$y_{n+1} = A(3)^{n+1} + B(-4)^{n+1} = 3A3^n - 4B(-4)^n$$
$$y_{n+2} = A(3)^{n+2} + B(-4)^{n+2} = 9A3^n + 16B(-4)^n$$
$$y_{n+2} + y_{n+1} - 12y_n = 0$$

Solve $a_k = 3a_{k-1}$, for $k \geq 1$, with $a_0 = 2$.

Solution:

$$Let\ G(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots \qquad \ldots(1)$$

where $G(x)$ is the generating function for the sequence $\{a_n\}$.

Given $a_k = 3a_{k-1} \Rightarrow 3a_{k-1} - a_k = 0$

Multiplying by x_k and summing from 1 to ∞, we have

$$3\sum_{k=1}^{\infty} a_{k-1}x^k - \sum_{k=1}^{\infty} a_k x^k = 0$$

$$3x\sum_{k=1}^{\infty} a_{k-1}x^{k-1} - \sum_{k=1}^{\infty} a_k x^k = 0$$

$3x(a_0 + a_1x + a_2x^2 + \cdots) - (a_1x + a_2x^2 + \cdots) = 0$

$3xG(x) - (G(x) - a_0) = 0 \qquad [from\ (1)]$

$G(x)(3x - 1) - 2 = 0$

$$G(x) = \frac{2}{(3x-1)} = -\frac{2}{1-3x}$$

$$\sum_{k=0}^{\infty} a_k x^k = -2\sum_{k=0}^{\infty} 3^k x^k \qquad \left[\because \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n\right]$$

$a_k =$ Coefficient of x^k in $G(x)$

$a_k = -2\,(3^n)$

If seven colours are used to paint 50 bicycles, then show that at least 8 bicycles will be the same colour.

Solution:

By Pigeon principle,

If there are n pigeons and k holes, then there is at least one hole contains at least $\left\lfloor\frac{k-1}{n}\right\rfloor + 1$ pigeons.

Here $k = 50, n = 7$

$$\left\lfloor\frac{k-1}{n}\right\rfloor + 1 = \left\lfloor\frac{50-1}{7}\right\rfloor + 1 = \left\lfloor\frac{49}{7}\right\rfloor + 1 = 7 + 1 = 8$$

∴There is at least 8 bicycles will be the same colour.

Solve the recurrence relation $y(n) - 8y(n-1) + 16y(n-2) = 0$ for $n \geq 2$, where $y(2) = 16$ and $y(3) = 80$.

Solution:

$$y(n) - 8y(n-1) + 16y(n-2) = 0 \ldots (1)$$

Let $y_n = r^n$ be the solution of (1).

$$(1) \Rightarrow r^n - 8r^{n-1} + 16r^{n-2} = 0$$

$$r^n \left[1 - \frac{8}{r} + \frac{16}{r^2} = 0\right]$$

$$\frac{r^n}{r^2}[r^2 - 8r + 16 = 0]$$

The characteristic equation is $r^2 - 8r + 16 = 0$

$$(r-4)^2 = 0 \Rightarrow r = 4,4$$

Hence the solution to this recurrence relation is

$$y_n = \alpha_1 4^n + \alpha_2 n 4^n \ldots (2)$$

$$y_2 = 16 \Rightarrow \alpha_1 + 2\alpha_2 = 1 \ldots (3)$$

$$y_3 = 80 \Rightarrow \alpha_1 + 3\alpha_2 = \frac{5}{4} \ldots (4)$$

Solving (3) and (4), we get

$$\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{4}$$

Substituting $\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{4}$ in (2),

$$y_n = \frac{1}{2}4^n + \frac{1}{4}n4^n$$

Find the number of non-negative integer solutions of the equation $x_1 + x_2 + x_3 = 11$.

Solution:

If there are r unknowns and their sum is n, then the number of non-negative integer solution for the problem is $(n+r-1)C_{r-1}$

Here there are 3 unknowns and the sum is 7

∴ The number of non-negative integer solutions of the equation $x_1 + x_2 + x_3 = 11$ is

$$(11+3-1)C_{3-1} = 13C_2 = 78$$

Find the recurrence relation for the Fibonacci sequence.

Solution:

$$f_n = f_{n-1} + f_{n-2}, n \geq 2 \text{ and } f_0 = 0, f_1 = 1$$

Use Mathematical induction show that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution:

$$Let\ P(n): 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \qquad \ldots (1)$$

$$P(1): 1^2 = \frac{1(1+1)(2+1)}{6}$$

$$1 = \frac{6}{6} \Rightarrow 1 = 1$$

∴ $P(1)$ is true.

Let us assume that $P(n)$ is true. Now we have to prove that $P(n+1)$ is true.
To prove:

$$P(n+1): 1^2+2^2+3^2+\cdots+(n+1)^2=\frac{(n+1)(n+2)(2n+3)}{6}$$

$$1^2+2^2+3^2+\cdots+n^2+(n+1)^2=\frac{n(n+1)(2n+1)}{6}+(n+1)^2 \qquad (from\ (1))$$

$$=\frac{n(n+1)(2n+1)+6(n+1)^2}{6}$$

$$=\frac{(n+1)[n(2n+1)+6(n+1)]}{6}$$

$$=\frac{(n+1)[2n^2+n+6n+6]}{6}$$

$$=\frac{(n+1)[2n^2+7n+6]}{6}$$

$$1^2+2^2+3^2+\cdots+n^2+(n+1)^2=\frac{(n+1)(n+2)(2n+3)}{6}$$

$\therefore P(n+1)$ is true.
$\therefore$ By induction method,

$$P(n): 1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}\ is\ true\ for\ all\ positive\ integers.$$

There are 2500 students in a college, of these 1700 have taken a course in C, 1000 have taken a course in Pascal and 550 have taken a course in Networking. Further 750 have taken courses in both C and Pascal. 400 have taken courses in both C and Networking, and 275 have taken courses in both Pascal and Networking. If 200 of these students have taken courses in C, Pascal and Networking.
(1) How many of these 2500 students have taken a course in any of these three courses C, Pascal and Networking?
(2) How many of these 2500 students have not taken a course in any of these three courses C, Pascal and Networking?
Solution:

Let U denote the number of students in a college.
Let A denote the number of students taken a course in C.
Let B denote the number of students taken a course in PASCAL.
Let C denote the number of students taken a course in Networking.
$|U|=2500, |A|=1700, |B|=1000, |C|=550, |A\cap B|=750, |A\cap C|=400,$
$|B\cap C|=275, |A\cap B\cap C|=200$
(1) The number of students has taken a course in any of these three courses C, Pascal and Networking
$We\ know\ that$
$|A\cup B\cup C|=|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$
$=1700+1000+550-750-400-275+200$
$|A\cup B\cup C|=2025.$
(2) The number of students has not taken a course in any of these three courses C, Pascal and Networking is
$|(A\cup B\cup C)'|=|U|-|(A\cup B\cup C)|=2500-2025=475$

Using generating function solve $\boldsymbol{y_{n+2}-5y_{n+1}+6y_n=0}$, $\boldsymbol{n\geq 0}$ with $\mathbf{y_o=1}$ and

$\mathbf{y_1 = 1}$.

Solution:

$Let\ G(x) = \sum_{n=0}^{\infty} y_n x^n$ ... (1) where $G(x)$ is the generating function for the sequence $\{y_n\}$.

Given $y_{n+2} - 5y_{n+1} + 6y_n = 0$

Multiplying by x_n and summing from 0 to ∞, we have

$$\sum_{n=0}^{\infty} y_{n+2}x^n - 5\sum_{n=0}^{\infty} y_{n+1}x^n + 6\sum_{n=0}^{\infty} y_n x^n = 0$$

$$\frac{1}{x^2}\sum_{n=0}^{\infty} y_{n+2}x^{n+2} - \frac{5}{x}\sum_{n=0}^{\infty} y_{n+1}x^{n+1} + 6\sum_{n=0}^{\infty} y_n x^n = 0$$

$$\frac{1}{x^2}(G(x) - y_1x - y_0) - \frac{5}{x}(G(x) - y_0) + 6G(x) = 0 \qquad [from\ (1)]$$

$$G(x)\left(\frac{1}{x^2} - \frac{5}{x} + 6\right) - \frac{y_1}{x} - \frac{y_0}{x^2} + \frac{5y_0}{x} = 0$$

$$G(x)\left(\frac{1}{x^2} - \frac{5}{x} + 6\right) - \frac{1}{x} - \frac{1}{x^2} + \frac{5}{x} = 0 \Rightarrow G(x)\left(\frac{6x^2 - 5x + 1}{x^2}\right) = \frac{1}{x^2} - \frac{4}{x}$$

$$G(x)\left(\frac{6x^2 - 5x + 1}{x^2}\right) = \frac{1 - 4x}{x^2}$$

$$G(x) = \frac{1 - 4x}{6x^2 - 5x + 1} = \frac{1 - 4x}{(3x - 1)(2x - 1)}$$

$$\frac{1 - 4x}{(3x - 1)(2x - 1)} = \frac{A}{(3x - 1)} + \frac{B}{(2x - 1)}$$

$1 - 4x = A(2x - 1) + B(3x - 1)$ ... (2)

Put $x = \frac{1}{2}$ in (2)

$$1 - 4\left(\frac{1}{2}\right) = B\left(\frac{3}{2} - 1\right) \Rightarrow \frac{1}{2}B = -1 \Rightarrow B = -2$$

Put $x = \frac{1}{3}$ in (2)

$$1 - 4\left(\frac{1}{3}\right) = A\left(\frac{2}{3} - 1\right) \Rightarrow -\frac{1}{3}A = -\frac{1}{3} \Rightarrow A = 1$$

$$G(x) = \frac{1}{(3x - 1)} - \frac{2}{(2x - 1)} = -\frac{1}{(1 - 3x)} + \frac{2}{(1 - 2x)}$$

$$\sum_{n=0}^{\infty} y_n x^n = -\sum_{n=0}^{\infty} 3^n x^n + 2\sum_{n=0}^{\infty} 2^n x^n$$

y_n = Coefficient of x^n in $G(x)$

$y_n = -3^n + 2^{n+1}$

A box contains six white balls and five red balls. Find the number of ways four balls can be drawn from the box if

(1) They can be any colour

(2) Two must be white and two red

(3) They must all be the same colour.

Solution:

Total number of balls $= 6 + 5 = 11$

(1) The number of ways four balls can be drawn from the box if they can be any colour is

$$11C_4 = \frac{11 \times 10 \times 9 \times 8}{4!} = 330$$

(2) The number of ways four balls can be drawn from the box if two must be white and two red

$$6C_2 \times 5C_2 = \frac{6 \times 5}{2!} \times \frac{5 \times 4}{2!} = 15 \times 10 = 150$$

(3) The number of ways four balls can be drawn from the box if they must all be the same colour.

$$6C_4 + 5C_4 = 6C_2 + 5C_1 = \frac{6 \times 5}{2!} + 5 = 15 + 5 = 20$$

Prove by the principle of Mathematical induction, for $'n'$ a positive integer

$$\mathbf{1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}}$$

Solution:

$$Let\ P(n): 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \qquad \ldots (1)$$

$$P(1): 1^2 = \frac{1(1+1)(2+1)}{6}$$

$$1 = \frac{6}{6} \Rightarrow 1 = 1$$

$\therefore P(1)$ is true.

Let us assume that $P(n)$ is true. Now we have to prove that $P(n+1)$ is true.

To prove:

$$P(n+1): 1^2 + 2^2 + 3^2 + \cdots + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \qquad (from\ (1))$$

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$$

$$= \frac{(n+1)[n(2n+1) + 6(n+1)]}{6}$$

$$= \frac{(n+1)[2n^2 + n + 6n + 6]}{6}$$

$$= \frac{(n+1)[2n^2 + 7n + 6]}{6}$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

$\therefore P(n+1)$ is true.

$\therefore$ By induction method,

$$P(n): 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}\ is\ true\ for\ all\ positive\ integers.$$

Find the number of distinct permutations that can be formed from all the letters of each word (1) RADAR (2) UNUSUAL.

Solution:

(1) The word RADAR contains 5 letters of which 2 A's and 2 R's are there.

$$The\ number\ of\ possible\ words\ = \frac{5!}{2!\,2!} = 30$$

Number of distinct permutation $= 30.$

(2) The word UNUSUAL contains 7 letters of which 3 U's are there.

$$The\ number\ of\ possible\ words\ = \frac{7!}{3!} = 840$$

Number of distinct permutation $= 840.$

Solve the recurrence relation, $(n) = S(n-1) + 2S(n-2)$, with $S(0) = 3, S(1) = 1$, by finding its generating function.

Solution:

The given recurrence relation is $2a_{n-2} + a_{n-1} - a_n = 0$ with $a_0 = 3, a_1 = 1$.

$$Let\ G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad \ldots(1)$$

where $G(x)$ is the generating function for the sequence $\{a_n\}$.

Given $2a_{n-2} + a_{n-1} - a_n = 0$

Multiplying by x_n and summing from 2 to ∞, we have

$$2\sum_{n=2}^{\infty} a_{n-2}x^n + \sum_{n=2}^{\infty} a_{n-1}x^n - \sum_{n=2}^{\infty} a_n x^n = 0$$

$$2x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2} + x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} - \sum_{n=2}^{\infty} a_n x^n = 0$$

$$2x^2(a_0 + a_1x + a_2x^2 + \cdots) + x(a_1x + a_2x^2 + \cdots) - (a_2x^2 + a_3x^3 + a_4x^4 + \cdots) = 0$$

$$2x^2G(x) + xG(x) - xa_0 - G(x) + a_0 + a_1x = 0 \qquad [from\ (1)]$$

$$G(x)(2x^2 + x - 1) - 3x + 3 + x = 0$$

$$G(x)(2x^2 + x - 1) = 2x - 3$$

$$G(x) = \frac{2x-3}{(2x^2+x-1)} = \frac{2x-3}{-(1+x)(1-2x)} = \frac{3-2x}{(1+x)(1-2x)}$$

$$\frac{3-2x}{(1+x)(1-2x)} = \frac{A}{1+x} + \frac{B}{(1-2x)}$$

$$3 - 2x = A(1-2x) + B(1+x) \ldots(2)$$

Put $x = \frac{1}{2}$ in (2)

$$3 - 1 = B\left(1 + \frac{1}{2}\right) \Rightarrow \frac{3}{2}B = 2 \Rightarrow B = \frac{4}{3}$$

Put $x = -1$ in (2)

$$3 + 2 = A(1+2) \Rightarrow 3A = 5 \Rightarrow A = \frac{5}{3}$$

$$G(x) = \frac{\frac{5}{3}}{1+x} + \frac{\frac{4}{3}}{(1-2x)}$$

$$\sum_{n=0}^{\infty} a_n x^n = \frac{5}{3}\sum_{n=0}^{\infty}(-1)^n\, x^n + \frac{4}{3}\sum_{n=0}^{\infty} 2^n x^n \qquad \left[\because \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n\right]$$

a_n = Coefficient of x^n in $G(x)$

$$a_n = \frac{5}{3}(-1)^n + \frac{4}{3}2^n$$

Prove, by mathematical induction, that for all $\boldsymbol{n \geq 1, n^3 + 2n}$ **is a multiple of 3.**

Solution:

$Let\ P(n): n \geq 1, n^3 + 2n$ is a multiple of 3. ... (1)

$P(1): 1^3 + 2(1) = 1 + 2 = 3$ is a multiple of 3.

$\therefore P(1)$ is true.

Let us assume that $P(n)$ is true. Now we have to prove that $P(n+1)$ is true.

To prove:

$$P(n+1): (n+1)^3 + 2(n+1) \text{ is a multiple of } 3$$

$$\begin{aligned}(n+1)^3 + 2(n+1) &= n^3 + 3n + 3n^2 + 1 + 2n + 2 \\ &= n^3 + 2n + 3n + 3n^2 + 3 \\ &= n^3 + 2n + 3(n^2 + n + 1)\end{aligned}$$

From (1) $n^3 + 2n$ is a multiple of 3

$$\therefore (n+1)^3 + 2(n+1)\ is\ a\ multiple\ of\ 3$$

$\therefore P(n+1)$ is true.

$\therefore$ By induction method,

$P(n): n \geq 1, n^3 + 2n$ is a multiple of 3, is true for all positive integer n.

Using the generating function, solve the difference equation

$$\mathbf{y_{n+2} - y_{n+1} - 6y_n = 0, y_1 = 1, y_0 = 2}$$

Solution:

$Let\ G(x) = \sum_{n=0}^{\infty} y_n x^n$ ... (1) where $G(x)$ is the generating function for the sequence $\{y_n\}$.

Given $y_{n+2} - y_{n+1} - 6y_n = 0$

Multiplying by x_n and summing from 0 to ∞, we have

$$\sum_{n=0}^{\infty} y_{n+2}x^n - \sum_{n=0}^{\infty} y_{n+1}x^n - 6\sum_{n=0}^{\infty} y_n x^n = 0$$

$$\frac{1}{x^2}\sum_{n=0}^{\infty} y_{n+2}x^{n+2} - \frac{1}{x}\sum_{n=0}^{\infty} y_{n+1}x^{n+1} - 6\sum_{n=0}^{\infty} y_n x^n = 0$$

$$\frac{1}{x^2}(G(x) - y_1 x - y_0) - \frac{1}{x}(G(x) - y_0) - 6G(x) = 0 \qquad [from\ (1)]$$

$$G(x)\left(\frac{1}{x^2} - \frac{1}{x} - 6\right) - \frac{y_1}{x} - \frac{y_0}{x^2} + \frac{y_0}{x} = 0$$

$$G(x)\left(\frac{1}{x^2} - \frac{1}{x} - 6\right) - \frac{1}{x} - \frac{2}{x^2} + \frac{2}{x} = 0 \Rightarrow G(x)\left(\frac{6x^2 - x + 1}{x^2}\right) = \frac{2}{x^2} - \frac{1}{x}$$

$$G(x)\left(\frac{1 - x - 6x^2}{x^2}\right) = \frac{2 - x}{x^2}$$

$$G(x) = \frac{2-x}{1-x-6x^2} = \frac{2-x}{(1-3x)(1+2x)}$$
$$\frac{2-x}{(1-3x)(1+2x)} = \frac{A}{(1-3x)} + \frac{B}{(1+2x)}$$
$$2-x = A(2x+1) + B(1-3x) \ldots (2)$$
Put $x = -\frac{1}{2}$ in (2)
$$2 - \left(-\frac{1}{2}\right) = B\left(1+\frac{3}{2}\right) \Rightarrow \frac{5}{2}B = \frac{5}{2} \Rightarrow B = 1$$
Put $x = \frac{1}{3}$ in (2)
$$2 - \left(\frac{1}{3}\right) = A\left(\frac{2}{3}+1\right) \Rightarrow \frac{5}{3}A = \frac{5}{3} \Rightarrow A = 1$$
$$G(x) = \frac{1}{(1-3x)} + \frac{1}{(1+2x)} = \frac{1}{(1-3x)} + \frac{1}{\left(1-(-2x)\right)}$$

$$\sum_{n=0}^{\infty} y_n x^n = \sum_{n=0}^{\infty} 3^n x^n + \sum_{n=0}^{\infty} (-2)^n x^n \qquad \left[\because \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n\right]$$
y_n = Coefficient of x^n in $G(x)$
$y_n = 3^n + (-2)^n$

How many positive integers n can be formed using the digits $3, 4, 4, 5, 5, 6, 7$ if n has to exceed 5000000?

Solution:

The positive integer n exceeds 5000000 if the first digit is either 5 or 6 or 7.
If the first digit is 5 then the remaining six digits are 3,4,4,5,6,7.
Then the number of positive integers formed by six digits is
$$\frac{6!}{2!} = 360 \qquad [Since\ 4\ appears\ twice]$$
If the first digit is 6 then the remaining six digits are 3,4,4,5,5,7.
Then the number of positive integers formed by six digits is
$$\frac{6!}{2!\,2!} = 180 \qquad [Since\ 4\ \&\ 5\ appears\ twice]$$
If the first digit is 7 then the remaining six digits are 3,4,4,5,6,5.
Then the number of positive integers formed by six digits is
$$\frac{6!}{2!\,2!} = 180 \qquad [Since\ 4\ \&\ 5\ appears\ twice]$$
∴ The number of positive integers n can be formed using the digits 3,4,4,5,5,6,7 if n has to exceed 5000000 is $360 + 180 + 180 = 720$.

Find the number of integers between 1 and 250 both inclusive that are divisible by any of the integers 2,3,5,7.

Solution:

Let A, B, C and D represents the integer from 1 to 250 that are divisible by 2,3,5 and 7 respectively.
$$|A| = \left\lfloor\frac{250}{2}\right\rfloor = 125, |B| = \left\lfloor\frac{250}{3}\right\rfloor = 83, |C| = \left\lfloor\frac{250}{5}\right\rfloor = 50, |D| = \left\lfloor\frac{250}{7}\right\rfloor = 35$$

$$|A\cap B| = \left\lfloor\frac{250}{2\times 3}\right\rfloor = 41, |A\cap C| = \left\lfloor\frac{250}{2\times 5}\right\rfloor = 25, |A\cap D| = \left\lfloor\frac{250}{2\times 7}\right\rfloor = 17, |B\cap C| = \left\lfloor\frac{250}{3\times 5}\right\rfloor = 16$$
$$|B\cap D| = \left\lfloor\frac{250}{3\times 7}\right\rfloor = 11, |C\cap D| = \left\lfloor\frac{250}{5\times 7}\right\rfloor = 7, |A\cap B\cap C| = \left\lfloor\frac{250}{2\times 3\times 5}\right\rfloor = 8$$
$$|A\cap B\cap D| = \left\lfloor\frac{250}{2\times 3\times 7}\right\rfloor = 5, |A\cap C\cap D| = \left\lfloor\frac{250}{2\times 5\times 7}\right\rfloor = 3, |B\cap C\cap D| = \left\lfloor\frac{250}{3\times 5\times 7}\right\rfloor = 2$$
$$|A\cap B\cap C\cap D| = \left\lfloor\frac{250}{2\times 3\times 5\times 7}\right\rfloor = 1$$

∴The number of integers between 1 and 250 both inclusive that are divisible by any of the integers 2,3,5,7 is

$$|A\cup B\cup C\cup D| = |A| + |B| + |C| + |D| - |A\cap B| - |A\cap C| - |A\cap D| - |B\cap C| - |B\cap D|$$
$$-|C\cap D| + |A\cap B\cap C| + |A\cap B\cap D| + |A\cap C\cap D| + |B\cap C\cap D| - |A\cap B\cap C\cap D|$$
$$|A\cup B\cup C\cup D| = 125 + 83 + 50 + 35 - 41 - 25 - 17 - 16 - 11 - 7 + 8 + 5 + 3 + 2 - 1$$
$$|A\cup B\cup C\cup D| = 193$$

Use mathematical induction to prove the inequality $n < 2^n$ for all positive integer n.

Proof:

$Let\ P(n): n < 2^n$ ... (1)

$P(1): 1 < 2^1$

$\Rightarrow 1 < 2$

∴ $P(1)$ is true.

Let us assume that $P(n)$ is true. Now we have to prove that $P(n+1)$ is true.

To prove:

$P(n+1): n+1 < 2^{n+1}$

$n < 2^n$ $(from\ (1))$

$$n+1 < 2^n + 1$$
$$n+1 < 2^n + 2^n \quad [\because 1 < 2^n]$$
$$n+1 < 2.2^n$$
$$n+1 < 2^{n+1}$$

∴ $P(n+1)$ is true.

∴ By induction method,

$$P(n): n < 2^n\ is\ true\ for\ all\ positive\ integers.$$

What is the maximum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade if there are five possible grades A, B, C, D and F.

Solution:

By Pigeonhole principle, If there are n holes and k pigeons $n \le k$ then there is at least one hole contains at least $\left\lfloor\frac{k-1}{n}\right\rfloor + 1$ pigeons.

Here $n = 5$

$$\left\lfloor\frac{k-1}{5}\right\rfloor + 1 = 6$$
$$\frac{k-1}{5} = 5 \Rightarrow k - 1 = 25 \Rightarrow k = 26$$

The maximum number of students required in a discrete mathematics class is 26.

Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty member from mathematics department and four from the computer science department?

Solution:
The number of ways to select 3 mathematics faculty members from 9 faculty members is $9C_3$ ways.
The number of ways to select 4 computer Science faculty members from 11 faculty members is $11C_4$ ways.
The number of ways to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty member from mathematics department and four from the computer science department is $9C_3.11C_4$ ways.

$$9C_3.11C_4 = \frac{9\times 8\times 7}{3!}.\frac{11\times 10\times 9\times 8}{4!} = 27720$$

Using method of generating function to solve the recurrence relation $a_n + 3a_{n-1} - 4a_{n-2} = 0; n \geq 2$, given that $a_0 = 3$ and $a_1 = -2$.

Solution:

$$Let\ G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad \ldots(1)$$

where $G(x)$ is the generating function for the sequence $\{a_n\}$.

Given $\quad a_n + 3a_{n-1} - 4a_{n-2} = 0$

Multiplying by x_n and summing from 2 to ∞, we have

$$\sum_{n=2}^{\infty} a_n x^n + 3\sum_{n=2}^{\infty} a_{n-1}x^n - 4\sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

$$\sum_{n=2}^{\infty} a_n x^n + 3x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} - 4x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2} = 0$$

$$(a_2x^2 + a_3x^3 + a_4x^4 + \cdots) + 3x(a_1x + a_2x^2 + \cdots) - 4x^2(a_0 + a_1x + a_2x^2 + \cdots) = 0$$

$$G(x) - a_0 - a_1x + 3xG(x) - 3xa_0 - 4x^2G(x) = 0 \qquad [from\ (1)]$$

$$G(x)(1 + 3x - 4x^2) - 3 + 2x - 9x = 0$$

$$G(x)(1 + 3x - 4x^2) = 3 - 7x$$

$$G(x) = \frac{3 - 7x}{(1 + 3x - 4x^2)} = \frac{3 - 7x}{(1 + 4x)(1 - x)}$$

$$\frac{3 - 7x}{(1 + 4x)(1 - x)} = \frac{A}{(1 + 4x)} + \frac{B}{(1 - x)}$$

$$3 - 7x = A(1 - x) + B(1 + 4x) \ldots(2)$$

Put $x = -\frac{1}{4}$ in (2)

$$3 - 7\left(-\frac{1}{4}\right) = A\left(1 + \frac{1}{4}\right) \Rightarrow \frac{5}{4}A = 3 + \frac{7}{4} \Rightarrow A = \frac{19}{5}$$

Put $x = 1$ in (2)

$$3 - 7 = B(1 + 4) \Rightarrow 5B = -4 \Rightarrow B = -\frac{4}{5}$$

$$G(x) = \frac{\frac{19}{5}}{(1 + 4x)} - \frac{\frac{4}{5}}{(1 - x)}$$

$$\sum_{n=0}^{\infty} a_n x^n = \frac{19}{5}\sum_{n=0}^{\infty}(-4)^n x^n - \frac{4}{5}\sum_{n=0}^{\infty} x^n \qquad \left[\because \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n\right]$$

$a_n =$ Coefficient of x^n in $G(x)$

$$a_n = \frac{19}{5}(-4)^n - \frac{4}{5}$$

Prove, by mathematical induction, that for all $\boldsymbol{n \geq 1, n^3 + 2n}$ **is a multiple of 3.**

Solution:

$Let\ P(n): n \geq 1, n^3 + 2n$ is a multiple of 3. ... (1)

$P(1): 1^3 + 2(1) = 1 + 2 = 3$ is a multiple of 3.

$\therefore P(1)$ is true.

Let us assume that $P(n)$ is true. Now we have to prove that $P(n+1)$ is true.

To prove:

$$P(n+1): (n+1)^3 + 2(n+1) \text{ is a multiple of } 3$$

$$(n+1)^3 + 2(n+1) = n^3 + 3n + 3n^2 + 1 + 2n + 2$$
$$= n^3 + 2n + 3n + 3n^2 + 3$$
$$= n^3 + 2n + 3(n^2 + n + 1)$$

From (1) $n^3 + 2n$ is a multiple of 3

$$\therefore (n+1)^3 + 2(n+1)\ is\ a\ multiple\ of\ 3$$

$\therefore P(n+1)$ is true.

$\therefore$ By induction method,

$P(n): n \geq 1, n^3 + 2n$ is a multiple of 3, is true for all positive integer n.

Using the generating function, solve the difference equation

$$\mathbf{y_{n+2} - y_{n+1} - 6y_n = 0, y_1 = 1, y_0 = 2}$$

Solution:

$Let\ G(x) = \sum_{n=0}^{\infty} y_n x^n$ ... (1) where $G(x)$ is the generating function for the sequence $\{y_n\}$.

Given $y_{n+2} - y_{n+1} - 6y_n = 0$

Multiplying by x_n and summing from 0 to ∞, we have

$$\sum_{n=0}^{\infty} y_{n+2}x^n - \sum_{n=0}^{\infty} y_{n+1}x^n - 6\sum_{n=0}^{\infty} y_n x^n = 0$$

$$\frac{1}{x^2}\sum_{n=0}^{\infty} y_{n+2}x^{n+2} - \frac{1}{x}\sum_{n=0}^{\infty} y_{n+1}x^{n+1} - 6\sum_{n=0}^{\infty} y_n x^n = 0$$

$$\frac{1}{x^2}(G(x) - y_1 x - y_0) - \frac{1}{x}(G(x) - y_0) - 6G(x) = 0 \qquad [from\ (1)]$$

$$G(x)\left(\frac{1}{x^2} - \frac{1}{x} - 6\right) - \frac{y_1}{x} - \frac{y_0}{x^2} + \frac{y_0}{x} = 0$$

$$G(x)\left(\frac{1}{x^2} - \frac{1}{x} - 6\right) - \frac{1}{x} - \frac{2}{x^2} + \frac{2}{x} = 0 \Rightarrow G(x)\left(\frac{6x^2 - x + 1}{x^2}\right) = \frac{2}{x^2} - \frac{1}{x}$$

$$G(x)\left(\frac{1 - x - 6x^2}{x^2}\right) = \frac{2 - x}{x^2}$$

$$G(x) = \frac{2-x}{1-x-6x^2} = \frac{2-x}{(1-3x)(1+2x)}$$
$$\frac{2-x}{(1-3x)(1+2x)} = \frac{A}{(1-3x)} + \frac{B}{(1+2x)}$$
$$2-x = A(2x+1) + B(1-3x) \ldots (2)$$
Put $x = -\frac{1}{2}$ in (2)
$$2-\left(-\frac{1}{2}\right) = B\left(1+\frac{3}{2}\right) \Rightarrow \frac{5}{2}B = \frac{5}{2} \Rightarrow B = 1$$
Put $x = \frac{1}{3}$ in (2)
$$2-\left(\frac{1}{3}\right) = A\left(\frac{2}{3}+1\right) \Rightarrow \frac{5}{3}A = \frac{5}{3} \Rightarrow A = 1$$
$$G(x) = \frac{1}{(1-3x)} + \frac{1}{(1+2x)} = \frac{1}{(1-3x)} + \frac{1}{\left(1-(-2x)\right)}$$

$$\sum_{n=0}^{\infty} y_n x^n = \sum_{n=0}^{\infty} 3^n x^n + \sum_{n=0}^{\infty} (-2)^n x^n \qquad \left[\because \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n\right]$$
y_n = Coefficient of x^n in $G(x)$
$y_n = 3^n + (-2)^n$

Using mathematical induction to show that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{1}, n \geq 2$.

Solution:
$$Let\ P(n): \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}, n \geq 2 \qquad \ldots (1)$$
$$P(2): \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1.7071 > 1.414 = \sqrt{2}.$$
$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}$$
$\therefore P(2)$ is true.
Let us assume that $P(n)$ is true. Now we have to prove that $P(n+1)$ is true.
To prove:
$$P(n+1): \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > \sqrt{n+1}$$
$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} = \sqrt{n} + \frac{1}{\sqrt{n+1}} \quad [from\ (1)]$$
$$= \frac{\sqrt{n}\sqrt{n+1}+1}{\sqrt{n+1}}$$
$$= \frac{\sqrt{n^2+n}+1}{\sqrt{n+1}}$$
$$> \frac{\sqrt{n^2}+1}{\sqrt{n+1}}$$
$$> \frac{n+1}{\sqrt{n+1}}$$

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}}>\sqrt{n+1}$$

$\therefore$ $P(n+1)$ is true.
$\therefore$ By induction method,
$P(n): \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}>\sqrt{n},$ is true for all positive integer $n \geq 2$.

Using method of generating function to solve the recurrence relation
$a_n = 4a_{n-1} - 4a_{n-2} + 4^n; n \geq 2$, given that $a_0 = 2$ and $a_1 = 8$.
Solution:

$$Let\ G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad \ldots(1)$$

where $G(x)$ is the generating function for the sequence $\{a_n\}$.
Given $\quad a_n = 4a_{n-1} - 4a_{n-2} + 4^n \Rightarrow 4a_{n-2} - 4a_{n-1} + a_n = 4^n$
Multiplying by x_n and summing from 2 to ∞, we have

$$4\sum_{n=2}^{\infty} a_{n-2}x^n - 4\sum_{n=2}^{\infty} a_{n-1}x^n + \sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} 4^n\, x^n$$

$$4x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2} - 4x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} + \sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} 4^n\, x^n$$

$$4x^2(a_0 + a_1x + a_2x^2 + \cdots) - 4x(a_1x + a_2x^2 + \cdots) + (a_2x^2 + a_3x^3 + a_4x^4 + \cdots) = ((4x)^2 + (4x)^3 + \cdots)$$

$$4x^2(a_0 + a_1x + a_2x^2 + \cdots) - 4x(a_0 + a_1x + a_2x^2 + \cdots) + 4xa_0 + (a_2x^2 + a_3x^3 + a_4x^4 + \cdots) = ((4x)^2 + (4x)^3 + \cdots)$$

$$4x^2G(x) - 4xG(x) + 4xa_0 + G(x) - a_0 - a_1x = 16x^2(1 + 4x + (4x)^2 + \cdots) \qquad [from\ (1)]$$

$$G(x)(4x^2 - 4x + 1) - 2 + 8x - 8x = \frac{16x^2}{1-4x}$$

$$G(x)(4x^2 - 4x + 1) = \frac{16x^2}{1-4x} + 2 \Rightarrow G(x)(1-2x)^2 = \frac{16x^2 + 2(1-4x)}{1-4x}$$

$$G(x)(1-2x)^2 = \frac{16x^2 + 2 - 8x}{1-4x}$$

$$G(x) = \frac{16x^2 + 2 - 8x}{(1-4x)(1-2x)^2}$$

$$\frac{16x^2 + 2 - 8x}{(1-4x)(1-2x)^2} = \frac{A}{(1-4x)} + \frac{B}{(1-2x)} + \frac{C}{(1-2x)^2}$$

$$16x^2 + 2 - 8x = A(1-2x)^2 + B(1-2x)(1-4x) + C(1-4x) \ldots(2)$$

Put $x = \frac{1}{2}$ in (2)

$$16\left(\frac{1}{4}\right) + 2 - 8\left(\frac{1}{2}\right) = C(1-\ 2) \Rightarrow -C = 2 \Rightarrow C = -2$$

Put $x = \frac{1}{4}$ in (2)

$$16\left(\frac{1}{16}\right)+2-8\left(\frac{1}{4}\right)=A\left(1-\frac{1}{2}\right)^2\Rightarrow\frac{1}{4}A=1\Rightarrow A=4$$

Put $x=0$ in (2)

$2=A+B+C\Rightarrow 4+B-2=2\Rightarrow B=0$

$$G(x)=\frac{4}{(1-4x)}+\frac{0}{(1-2x)}-\frac{2}{(1-2x)^2}$$

$$\sum_{n=0}^{\infty}a_nx^n=4\sum_{n=0}^{\infty}4^nx^n-2\sum_{n=0}^{\infty}(n+1)2^nx^n$$

$$\left[\because\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n\ \ \&\ \frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}(n+1)x^n\right]$$

a_n = Coefficient of x^n in $G(x)$

$a_n=4^{n+1}-(n+1)2^{n+1}$