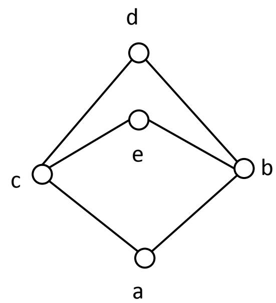
Unit – V Lattice and Boolean Algebra

The following is the hasse diagram of a partially ordered set. Verify whether it is a lattice.



Solution:

d and e are the upper bounds of c and b. As d and e cannot be compared, therefore the $LUB\ \{c, b\}$ does not exists. The Hasse diagram is not a lattice.

Give an example of a relation which is symmetric, transitive but not reflexive on $\{a, b, c\}$

Solution:

$R = \{(a, a), (a, b), (a, c), (b, a), (b, b), (c, a)\}$

Define partially ordered set.

A Set with a partially ordering relation is called a poset or partially ordered set.

Find the Partition of $A = \{0, 1, 2, 3, 4, 5\}$ with minsets generated by $B_1 = \{0, 2, 4\}$ and $B_2 = \{1, 5\}$.

Solution:

$B_1 \cap B_2 = \emptyset, B_1 \cup B_2 = \{0, 1, 2, 4, 5\} \neq A$, $(B_1 \cup B_2)' = \{3\}$

$B_1 \cup B_2 \cup (B_1 \cap B_2)' = \{0, 1, 2, 3, 4, 5\} = A$

Partition of $A = \{\{0, 2, 4\}, \{1, 5\}, \{3\}\}$

If a poset has a least element, then prove it is unique.

Proof:

Let $\langle L, \leq \rangle$ be a poset with a_1, a_2 be two least elements.

If a_1 is the least element, $a_1 \leq a_2$

If a_2 is the least element $a_2 \leq a_1$

By antisymmetric property $a_1 = a_2$

So that least element is unique.

If $R = \{(1, 1), (1, 2), (2, 3)\}$ and $S = \{(2, 1), (2, 2), (3, 2)\}$ are the relations on the set $A = \{1, 2, 3\}$. Verify whether $RoS = SoR$ by finding the relation matrices of RoS and SoR.

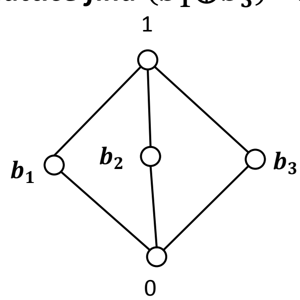
Solution:

$$M_R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, M_s = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$M_{RoS} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } M_{SoR} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$M_{RoS} \neq M_{SoR} \Rightarrow RoS \neq SoR$

In the following lattice find $(\boldsymbol{b_1} \oplus \boldsymbol{b_3}) * \boldsymbol{b_2}$



Solution:

$b_1 \oplus b_3 = 1$. Hence $(b_1 \oplus b_3) * b_2 = 1 * b_2 = b_2$

If $\boldsymbol{A_2} = \{\{\mathbf{1}, \mathbf{2}\}, \{\mathbf{3}\}\}, \boldsymbol{A_2} = \{\{\mathbf{1}\}, \{\mathbf{2}, \mathbf{3}\}\}$ ***and*** $\boldsymbol{A_3} = \{\{\mathbf{1}, \mathbf{2}, \mathbf{3}\}\}$ ***then show that*** $\boldsymbol{A_1}, \boldsymbol{A_2}$ ***and*** $\boldsymbol{A_3}$ ***are mutually disjoint.***

Solution:

$A_1 \cap A_2 = \emptyset, A_1 \cap A_3 = \emptyset, A_2 \cap A_3 = \emptyset$

Hence A_1, A_2 and A_3 are mutually disjoint.

Let $\boldsymbol{x} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$. ***If***

$\boldsymbol{R} = \{< x, y > \mid \boldsymbol{x} \in \boldsymbol{X} \wedge \boldsymbol{y} \in \boldsymbol{X} \wedge (\boldsymbol{x} - \boldsymbol{y}) \textbf{\textit{is an nonzero multiple of }} \mathbf{2}\}$

$\boldsymbol{S} = \{< x, y > \mid x \in \boldsymbol{X} \wedge \boldsymbol{y} \in \boldsymbol{X} \wedge (\boldsymbol{x} - \boldsymbol{y}) \textbf{\textit{is an nonzero multiple of }} \mathbf{3}\}$

Find $\boldsymbol{R} \cup \boldsymbol{S}$ ***and*** $\boldsymbol{R} \cap \boldsymbol{S}$.

Solution:

$R = \{(1,3), (3,1), (2,4), (4,2)\}, S = \{(1,4), (4,1)\}$

$R \cup S = \{(1,3), (3,1), (2,4), (4,2), (1,4), (4,1)\}, R \cap S = \emptyset$

$R \cap S = \{< x, y > \mid x \in X \wedge y \in X \wedge (x - y) \text{is an nonzero multiple of } 6\}$

If R and S are reflexive relations on a set A, then show that $\boldsymbol{R} \cup \boldsymbol{S}$ ***and*** $\boldsymbol{R} \cap \boldsymbol{S}$ ***are also reflexive relations on A.***

Solution:

Let $a \in A$. Since R and S are reflexive.

We have $(a, a) \in R$ and $(a, a) \in S \Rightarrow (a, a) \in R \cap S$

Hence $R \cap S$ is reflexive.

$(a, a) \in R$ or $(a, a) \in S \Rightarrow (a, a) \in R \cup S$

Hence $R \cup S$ is reflexive.

Define Equivalence relation. Give an example

Solution:

A relation R in a set A is called an equivalence relation if it is reflexive, symmetric and transitive.

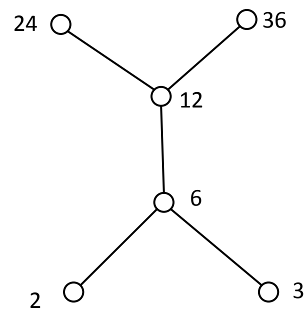
Eg: i) Equality of numbers on a set of real numbers

ii) Relation of lines being parallel on a set of lines in a plane.

Let $X = \{2, 3, 6, 12, 24, 36\}$ ***and the relation be such that*** $x \leq y$ iff x $divides$ y***. Draw the Hasse Diagram of*** $\langle X, \leq \rangle$.

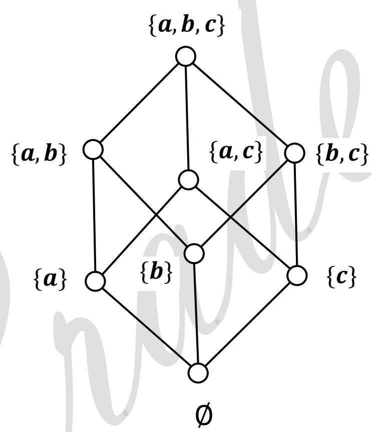
Solution:

The Hasse diagram is



Let A ***be a given finite set and*** $P(A)$ ***its power set. Let*** $\subseteq$ ***be the inclusion relation on the elements of*** $P(A)$***. Draw Hasse diagram of*** $\langle P(A), \leq \rangle$ ***for*** $A = \{a, b, c\}$

Solution:



Write the representing each of the relations from $\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \end{pmatrix}$

Solution:

Let $A = \{1, 2, 3\}$ and R be the relation defined on A corresponding to the given matrix. $\therefore$ $R = \{(1,1), (1,2), (1,3), (2,1), (2,3), (3,1), (3,2), (3,3)\}$

Which elements of the poset $[\{2, 4, 5, 10, 12, 20, 25\}, / \,]$ ***are maximal and which are minimal?***

(or)

Give an example for a poset that have more than one maximal element and more than one minimal element.

Solution:

$A = [\{2, 4, 5, 10, 12, 20, 25\},\ /\ ], / \; is\ the\ division\ relation.$

The maximal elements are 12, 20, 25 and the minimal elements are 2,5 .

Define Lattice

A Lattice in a partially ordered set $\langle \boldsymbol{L}, \leq \rangle$ in which every pair of elements $a, b \in L$ has the greatest lower bound and a least upper bound.

Let $\langle \boldsymbol{L}, \leq \rangle$ be a lattice. For any $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \boldsymbol{L}$ we have $\boldsymbol{a} * \boldsymbol{a} = \boldsymbol{a}$

Solution:

Since $a \leq a$, a is a lower bound of $\{a\}$. If b is any lower bound of $\{a\}$, then we have $b \leq a$. Thus we have $a \leq a$ or $b \leq a$ equivalently, a is an lower bound for $\{a\}$and any other lower bound of $\{a\}$ is smaller than a. This shows that a is the greatest lower bound of $\{a\}$, i.e., $GLB\{a, a\} = a$

$\therefore a * a = GLB\{a, a\} = a$

Define sublattice

Let $\langle L,*, \oplus \rangle$ be a lattice and let $S \subseteq L$ be a subset of L. Then $\langle S,*, \oplus \rangle$ is a sublattice of $\langle L,*, \oplus \rangle$ iff S is closed under both operations $*$ and $\oplus$.

Define Lattice Homomorphism

Let $\langle L,*, \oplus \rangle$ and $\langle S, \wedge, \vee \rangle$ be two lattices. A mapping $g: L \rightarrow S$ is called a lattice homomorphism from the lattice $\langle L,*, \oplus \rangle$to $\langle S, \wedge, \vee \rangle$ if for any $a, b \in L$

$g(a * b) = g(a) \wedge g(b)\ and\ g(a \oplus b) = g(a) \vee g(b)$

Define Modular

A lattice $\langle L,*, \oplus \rangle$ is called modular if for all $x, y, z \in L$

$$x \leq z \Rightarrow x \oplus (y * z) = (x \oplus y) * z$$

Define Distributive lattice.

A Lattice $\langle L,*, \oplus \rangle$is called a distributive lattice if for any $a, b, c \in L$

$a * (b \oplus c) = (a * b) \oplus (a * c)$

$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$

Prove that every distributive lattice is modular.

Proof:

Let $\langle L,*, \oplus \rangle$ be a distributive lattice.

$\forall a, b, c \in L$ we have , $a \oplus (b * c) = (a \oplus b) * (a \oplus c)$ ... (1)

Thus if $a \leq c$ then $a \oplus c = c$ ... (2)

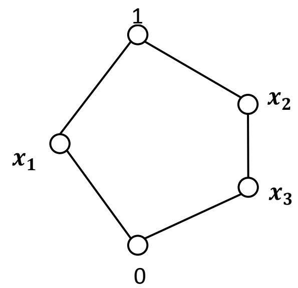
from (1) and (2) we get

$a \oplus (b * c) = (a \oplus b) * c$

So if $a * c$,then $a \oplus (b * c) = (a \oplus b) * c$.

$\therefore L$ is modular.

The lattice with the following Hasse diagram is not distributive and not modular.



Solution:

In this case, $(x_1 \oplus x_3) * x_2 = 1 * x_2 = x_2 \ldots (1)$

And $(x_1 * x_2) \oplus (x_3 * x_2) = 0 \oplus x_3 = x_3 \ldots (2)$

From (1) and (2) we get

$(x_1 \oplus x_3) * x_2 \neq (x_1 * x_2) \oplus (x_3 * x_2)$

Hence the lattice is not distributive.

$x_3 < x_2 \Rightarrow x_3 \oplus (x_1 * x_2) = x_3 \oplus 0 = x_3 \ldots (3)$

$(x_3 \oplus x_1) * x_2 = 1 * x_2 = x_2 \ldots (4)$

From (3) and (4) we get

$x_3 \oplus (x_1 * x_2) \neq (x_3 \oplus x_1) * x_2$

Hence the lattice is not modular.

PART-B

In a Lattice, show that $a = b$ and $c = d \Rightarrow a * c = b * d$

Solution:

For any $a, b, c \in L$

If $a = b \Rightarrow c * a \leq c * b$

$$\Rightarrow a * c \leq b * c \ldots (1)(By\ Commutative\ law)$$

For any $b, c, d \in L$

If $c = d \Rightarrow b * c \leq b * d \ldots (2)$

From (1) and (2) we get

$$a * c = b * d$$

In a distributive Lattice prove that

$a * b = a * c$ and $a \oplus b = a \oplus c \Rightarrow b = c$.

Solution:

$$(a * b) \oplus c = (a * c) \oplus c = c \ldots (1)\ [a * b = a * c\ \ and\ absorbtion\ law]$$

$$(a * b) \oplus c = (a \oplus c) * (b \oplus c)\ [Distributive\ law]$$

$$= (a \oplus b) * (b \oplus c) = (a \oplus b) * (c \oplus b)\ [a \oplus b = a \oplus c\ and\ commutative\ law]$$

$$= (a * c) \oplus b = (a * b) \oplus b = b \ldots (2)[Distributive\ and\ absorbtion\ law]$$

From (1) and (2) we get,

$$b = c$$

Establish De Morgan's laws in a Boolean algebra

Solution: Let $a, b \in (B, *, \oplus, ', 0, 1)$

To prove $(a \oplus b)' = a' * b'$

$$(a \oplus b) * (a' * b') = \big(a * (a' * b')\big) \oplus \big(b * (a' * b')\big)$$
$$= \big(a * (a' * b')\big) \oplus \big((a' * b') * b\big)$$
$$= \big((a * a') * b'\big) \oplus \big(a' * (b' * b)\big)$$
$$= (0 * b') \oplus (a' * 0) = 0 \oplus 0$$
$$(a \oplus b) * (a' * b') = 0 \ldots (1)$$
$$(a \oplus b) \oplus (a' * b') = \big((a \oplus b) \oplus a'\big) * \big((a \oplus b) \oplus b'\big)$$
$$= \big((b \oplus a) \oplus a'\big) * \big((a \oplus b) \oplus b'\big)$$
$$= \big(b \oplus (a \oplus a')\big) * \big(a \oplus (b \oplus b')\big)$$
$$= (b \oplus 1) * (a \oplus 1) = 1 * 1$$
$$(a \oplus b) \oplus (a' * b') = 1 \ldots (2)$$

From (1) *and* (2) *we get*,

$$\therefore (a \oplus b)' = a' * b'$$

To prove $(a * b)' = a' \oplus b'$

$$(a * b) \oplus (a' \oplus b') = \big(a \oplus (a' \oplus b')\big) * \big(b \oplus (a' \oplus b')\big)$$
$$= \big(a \oplus (a' \oplus b')\big) * \big((a' \oplus b') \oplus b\big)$$
$$= \big((a \oplus a') \oplus b'\big) * \big(a' \oplus (b' \oplus b)\big)$$
$$= (1 \oplus b') * (a' \oplus 1) = 1 * 1$$
$$(a * b) \oplus (a' \oplus b') = 1 \ldots (3)$$
$$(a * b) * (a' \oplus b') = \big((a * b) * a'\big) \oplus \big((a * b) * b'\big)$$
$$= \big((b * a) * a'\big) \oplus \big((a * b) * b'\big)$$
$$= \big(b * (a * a')\big) \oplus \big(a * (b * b')\big)$$
$$= (b * 0) \oplus (a * 0) = 0 \oplus 0$$
$$(a * b) * (a' \oplus b') = 0 \ldots (4)$$

From (3) *and* (4) *we get*,

$$(a * b)' = a' \oplus b'$$

In a Boolean algebra L, Prove that $(a \wedge b)' = a' \vee b', \forall a, b \in L$

Solution:

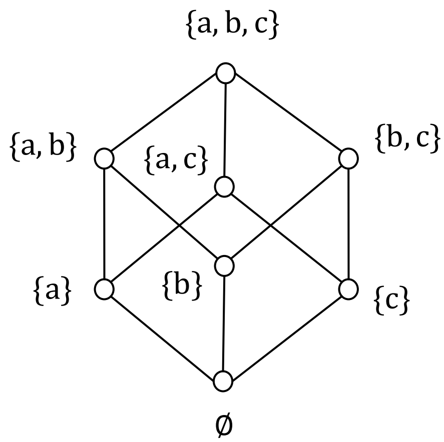
$$(a \wedge b) \vee (a' \vee b') = \big(a \vee (a' \vee b')\big) \wedge \big(b \vee (a' \vee b')\big)$$
$$= \big(a \vee (a' \vee b')\big) \wedge \big((a' \vee b') \vee b\big)$$
$$= \big((a \vee a') \vee b'\big) \wedge \big(a' \vee (b' \vee b)\big)$$
$$= (1 \vee b') \wedge (a' \vee 1) = 1 \wedge 1$$
$$(a \wedge b) \vee (a' \vee b') = 1 \ldots (1)$$
$$(a \wedge b) \wedge (a' \vee b') = \big((a \wedge b) \wedge a'\big) \vee \big((a \wedge b) \wedge b'\big)$$
$$= \big((b \wedge a) \wedge a'\big) \vee \big((a \wedge b) \wedge b'\big)$$
$$= \big(b \wedge (a \wedge a')\big) \vee \big(a \wedge (b \wedge b')\big)$$
$$= (b \wedge 0) \vee (a \wedge 0) = 0 \vee 0$$
$$(a \wedge b) \wedge (a' \vee b') = 0 \ldots (2)$$

From (1) *and* (2) *we get*,

$$(a \wedge b)' = a' \vee b'$$

Draw the Hasse diagram of the lattice L of all subsets of a, b, c under intersection and union.

Solution:



Define the relation P on $\{1, 2, 3, 4\}$ by $P = \{(a, b) / |a - b| = 1\}$. Determine the adjacency matrix of P^2

Solution:

$$P = \{(1,2), (2,1), (2,3), (3,2), (3,4), (4,3)\}.$$

$$M_P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$M_{P^2} = M_{PoP} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Let $(L, \leq)$ be a lattice. For any $a, b, c \in L$ if $b \leq c \Rightarrow a * b \leq a * c$

$$\boldsymbol{and\ \ a \oplus b \leq a \oplus c}$$

Solution:

$(a * b) * (a * c) = a * (b * a) * c = a * (a * b) * c$

$= (a * a) * (b * c) = a * b$

$$\therefore (a * b) * (a * c) = a * b$$

$$a * b \leq a * c$$

$(a \oplus b) * (a \oplus c) = a \oplus (b * c) = a \oplus c$

$$\therefore a \oplus b \leq a \oplus c$$

In a distributice lattice, show that

$(\boldsymbol{a * b}) \oplus (\boldsymbol{b * c}) \oplus (\boldsymbol{c * a}) = (\boldsymbol{a \oplus b}) * (\boldsymbol{b \oplus c}) * (\boldsymbol{c \oplus a})$

Solution:

$$(a * b) \oplus (b * c) \oplus (c * a) = (a * b) \oplus (c * b) \oplus (c * a)$$

$$= \big((a \oplus c) * b\big) \oplus (c * a)$$

$$= \left(\left((a \oplus c) * b\right) \oplus c\right) * \left(\left((a \oplus c) * b\right) \oplus a\right)$$
$$= \left(\left((a \oplus c) \oplus c\right) * (b \oplus c)\right) * \left(\left((a \oplus c) \oplus a\right) * (b \oplus a)\right)$$
$$= \left(\left((a \oplus c) \oplus c\right) * (b \oplus c)\right) * \left(\left(a \oplus (a \oplus c)\right) * (b \oplus a)\right)$$
$$= \left(\left(a \oplus (c \oplus c)\right) * (b \oplus c)\right) * \left(\left((a \oplus a) \oplus c\right) * (b \oplus a)\right)$$
$$= (a \oplus c) * (b \oplus c) * (a \oplus c) * (b \oplus a)$$
$$= (c \oplus a) * (b \oplus c) * (c \oplus a) * (a \oplus b)$$
$$= (b \oplus c) * (c \oplus a) * (c \oplus a) * (a \oplus b)$$
$$= (b \oplus c) * (c \oplus a) * (a \oplus b)$$
$$= (b \oplus c) * (a \oplus b) * (c \oplus a)$$
$$= (a \oplus b) * (b \oplus c) * (c \oplus a)$$

Simplify the Boolean expression $((\boldsymbol{x_1} + \boldsymbol{x_2}) + (\boldsymbol{x_1} + \boldsymbol{x_3}))\,.\,\boldsymbol{x_1}.\overline{\boldsymbol{x_2}}$

Solution:

$$\left((x_1 + x_2) + (x_1 + x_3)\right).x_1.\overline{x_2} = (x_1 + x_2).x_1.\overline{x_2} + (x_1 + x_3).x_1.\overline{x_2}$$
$$= x_1.x_1.\overline{x_2} + x_2.x_1.\overline{x_2} + x_1.x_1.\overline{x_2} + x_3.x_1.\overline{x_2}$$
$$= x_1.x_1.\overline{x_2} + x_1.x_2.\overline{x_2} + x_3.x_1.\overline{x_2}$$
$$= x_1.\overline{x_2} + x_1.0 + x_3.x_1.\overline{x_2}$$
$$= x_1.\overline{x_2} + x_3.x_1.\overline{x_2}$$
$$= x_1.\overline{x_2}$$

State and prove the distributive inequalities of a lattice.

Solution:

Let $(L, \leq)$ be a lattice. For any $a, b, c \in L$

I) $a * (b \oplus c) \geq (a * b) \oplus (a * c)$

II) $a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$

To prove $a * (b \oplus c) \geq (a * b) \oplus (a * c)$

From $a \geq a * b \text{ and } a \geq a * c \Rightarrow a \geq (a * b) \oplus (a * c) \ldots (1)$

$$b \oplus c \geq b \geq (a * b) \ldots (2)$$
$$b \oplus c \geq c \geq (a * c) \ldots (3)$$

From (2) and (3) we get,

$$b \oplus c \geq (a * b) \oplus (a * c) \ldots (4)$$

From (1) and (4) we get,

$$a * (b \oplus c) \geq (a * b) \oplus (a * c)$$

To prove $a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$

From $a \oplus b \geq a \text{ and } a \oplus c \geq a \Rightarrow (a \oplus b) * (a \oplus c) \geq a \ldots (5)$

$$b * c \leq b \leq (a \oplus b) \ldots (6)$$
$$b * c \leq c \leq (a \oplus c) \ldots (7)$$

From (6) and (7) we get,

$$b * c \leq (a \oplus b) * (a \oplus c) \ldots (8)$$

From (5) and (8) we get,

$$a * (b \oplus c) \geq (a * b) \oplus (a * c)$$
$$a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$$

In a lattice show that $a \leq b \Leftrightarrow a * b = a \Leftrightarrow a \oplus b = b$

Solution:

To prove $a \leq b \Leftrightarrow a * b = a$

Let us assume that $a \leq b$, we know that $a \leq a . \therefore a \leq a * b \ldots (1)$

From the definition we know that $a * b \leq a \ldots (2)$

From (1) and (2) we get $a * b = a$

$$\therefore a \leq b \Rightarrow a * b = a \ldots (I)$$

Now assume that $a * b = a$ but it is possible iff $a \leq b$

$$\therefore a * b = a \Rightarrow a \leq b \ldots (II)$$

From (I) and (II) we get

$$a \leq b \Leftrightarrow a * b = a$$

To prove $a * b = a \Leftrightarrow a \oplus b = b$

Let us assume that $a * b = a$

$$b \oplus (a * b) = b \oplus a = a \oplus b \ldots (3)$$

$$b \oplus (a * b) = b \ldots (4)$$

From (3) and (4) we get $a \oplus b = b$

$$\therefore a * b = a \Rightarrow a \oplus b = b \ldots (III)$$

Let us assume that $a \oplus b = b$

$$a * (a \oplus b) = a * b \ldots (5)$$

$$a * (a \oplus b) = a \ldots (6)$$

From (5) and (6) we get $a * b = a$

$$\therefore a \oplus b = b \Rightarrow a * b = a \ldots (IV)$$

From (III) and (IV) we get $a * b = a \Leftrightarrow a \oplus b = b$

Prove that every chain is a distributive lattice.

Solution:

Let $(L, \leq)$ be a chain and $a, b, c \in L$. Consider the following cases:

(I) $a \leq b \text{ and } a \leq c$, and (II) $a \geq b \text{ and } a \geq c$

For (I)

$$a * (b \oplus c) = a \ldots (1)$$

$$(a * b) \oplus (a * c) = a \oplus a = a \ldots (2)$$

For (II)

$$a * (b \oplus c) = b \oplus c \ldots (3)$$

$$(a * b) \oplus (a * c) = b \oplus c \ldots (4)$$

∴From (1),(2) and (3),(4)

$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

∴Every chain is a distributive lattice

Show that every distributive lattice is a modular. Whether the converse is true? Justify your answer

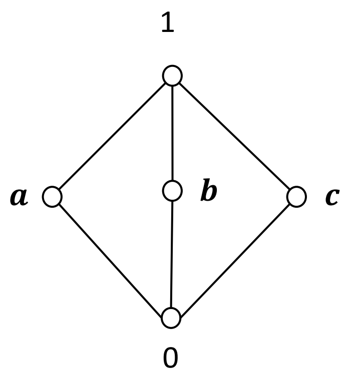
Solution:

Let $a, b, c \in L$ and assume that $a \leq c, then$

$$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$$
$$= (a \oplus b) * c$$

∴Every distributive lattice is modular.

For example let us consider the following lattice



Here in this lattice

$$\forall a, b, c \in L, a \leq b \Rightarrow a \oplus (b * c) = (a \oplus b) * c$$

∴The above lattice is modular.

$$a * (b \oplus c) = a * 1 = a \dots (1)$$
$$(a * b) \oplus (a * c) = 0 \oplus 0 = 0 \dots (2)$$

From (1) and (2) we get $a * (b \oplus c) \neq (a * b) \oplus (a * c)$

∴The above lattice is not distributive.

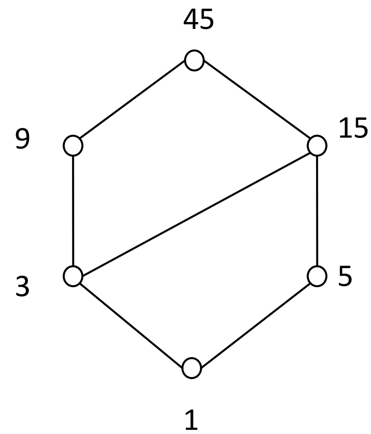
∴ Every distributive lattice is a modular but its converse is not true.

Find the sub lattices of $(D_{45}, /)$. Find its complement element.

Solution:

$$D_{45} = \{1, 3, 5, 9, 15, 45\} \; under \; division \; rule$$
$$1 \oplus 45 = 45 \; and \; 1 * 45 = 1$$
$$\therefore Complement \; of \; 1 \; is \; 45$$
$$5 \oplus 9 = 45 \; and \; 5 * 9 = 1$$
$$\therefore Complement \; of \; 5 \; is \; 9$$
$$3 \oplus 15 = 15 \; and \; 3 * 15 = 3$$
$$\therefore 3 \; and \; 15 \; has \; no \; Complement$$

∴ $(D_{45}, /)$ is not a complement lattice



The sub lattices of $(D_{45}, /)$ are given below

$$S_1 = \{1,3,5,9,15,45\},\ S_2 = \{1,3,9,45\}, S_3 = \{1,5,15,45\},$$
$$S_4 = \{1,3,5,15\}, S_5 = \{ 3,9,15,45\}, S_6 = \{1,3,9,15,45\},$$
$$S_7 = \{1,3,5,15,45\}, S_8 = \{1,3\}, S_9 = \{1,5\}, S_{10} = \{1,3,9\}, S_{11} = \{1,5,15\}$$
$$S_{12} = \{ 3,9,45\}, S_{13} = \{ 5,15,45\}, S_{14} = \{ 3,9\}, S_{15} = \{5,15\}, S_{16} = \{ 15,45\}$$
$$S_{17} = \{ 9,45\}, S_{18} = \{ 3,15\}, S_{19} = \{3,5,9,15,45\}$$

In any Boolean algebra, show that $\boldsymbol{a = b \Leftrightarrow ab' + a'b = 0}$

Proof:

Case i) To prove $a = b \Rightarrow ab' + a'b = 0$

$$ab' = bb' = 0 \ldots (1)[a = b \; and \; Complement \; law]$$
$$a'b = a'a = 0 \ldots (2)[a = b \; and \; Complement \; law]$$
$$ab' + a'b = 0 + 0 = 0 \quad [from \; (1) \; and \; (2)]$$

Case ii) To prove $ab' + a'b = 0 \Rightarrow a = b \ldots (3)$

$$ab' + a'b = 0$$
$$a + ab' + a'b = a + 0 \quad [b = c \Rightarrow a + b = a + c]$$
$$a + a'b = a \; [Absorbtion \; law \; and \; a + 0 = a]$$
$$(a + a')(a + b) = a[Distributive \; law]$$
$$1(a + b) = a \Rightarrow a + b = a \ldots (4)[Complement \; law]$$
$$Similarly \; from \; (3), we \; get \; ab' + a'b + b = 0 + b$$
$$[b = c \Rightarrow b + a = c + a]$$
$$ab' + b = b \; [Absorbtion \; law \; and \; 0 + b = b]$$
$$(a + b)(b' + b) = b[Distributive \; law]$$
$$(a + b)1 = b \Rightarrow a + b = b \ldots (5)[Complement \; law]$$

From (4) and (5) we get

$$a = b$$

Let $\boldsymbol{(L, \leq)}$ be a lattice. For any $\boldsymbol{a, b, c \in L}$ the following holds,

$$\boldsymbol{a \leq c \Leftrightarrow a \oplus (b * c) \leq (a \oplus b) * c}$$

Solution: To prove $a \leq c \Rightarrow a \oplus (b * c) \leq (a \oplus b) * c$

Let us assume that $a \leq c$,

$$a \oplus (b * c) \leq (a \oplus b) * (a \oplus c) \; [Distributive \; inequality]$$

$$\leq (a \oplus b) * c \; [Distributive\ \ inequality]$$

To prove $a \oplus (b * c) \;\leq\; (a \oplus b) * c \;\Rightarrow\; a \;\leq\; c$

Let us assume that $a \oplus (b * c) \;\leq\; (a \oplus b) * c$

$$(a \oplus b) * (a \oplus c\,) \;\leq\; (a \oplus b) * c [Distributive\ \ law]$$

$$\Rightarrow (a \oplus c\,) \leq c \ldots (1) \left[a * b \leq a * c \Rightarrow b \leq c\right]$$

$$a \oplus (b * c) \;\leq\; (a \oplus b) * c$$

$$a \oplus (b * c) \;\leq\; (a * c) \oplus (b * c) [Distributive\ \ law]$$

$$\Rightarrow a \leq (a * c) \leq (a \oplus c\,) \leq c \; [Definition\ of\ * and \oplus\ and (1)]$$

$$\Rightarrow a \;\leq\; c$$

Prove that the direct product of any two distributive lattices is a distributive lattice.

Solution:

Let $(L,*,\oplus)$ and $(S,\wedge,\vee)$ $be\ two\ distributive\ lattices\ and\ let\ (L \times S, ., +)$
$be\ the\ direct\ product\ of\ two\ lattices.$

For any $(a_1, b_1), (a_2, b_2)$ and $(a_3, b_3) \in L \times S$

$$(a_1, b_1).\big((a_2, b_2) + (a_3, b_3)\big) = (a_1, b_1).(a_2 \oplus a_3, b_2 \vee b_3)$$

$$= \big(a_1 * (a_2 \oplus a_3), b_1 \wedge (b_2 \vee b_3)\big)$$

$$= \big((a_1 * a_2) \oplus (a_1 * a_3), (b_1 \wedge b_2) \vee (b_1 \wedge b_3)\big)$$

$$= (a_1, b_1).(a_2, b_2) + (a_1, b_1).(a_3, b_3)$$

∴The direct product of any two distributive lattices is a distributive lattice.

Find the complement of every element of the lattice $<\ S_n\,, D\ >$ for $n = 75$.

Solution:

$$S_{45} = \{1, 3, 5, 15, 25, 75\}\ under\ division\ rule$$

$$1 \oplus 75 = 75\ and\ 1 * 75 = 1$$

$$\therefore Complement\ of\ 1\ is\ 75$$

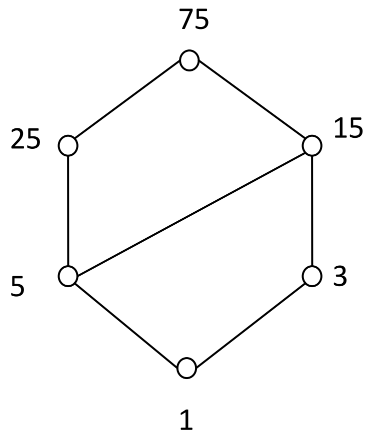
$$3 \oplus 25 = 75\ and\ 3 * 25 = 1$$

$$\therefore Complement\ of\ 3\ is\ 25$$

$$5 \oplus 15 = 15\ and\ 5 * 15 = 5$$

$$\therefore 5\ and\ 15\ has\ no\ Complement$$

∴ It is not a complement lattice



Write the Lattices of $(D_{35},/)$. Find its complements

Solution:

$$D_{35} = \{1, 5, 7, 35\}\ under\ division\ rule$$
$$1 \oplus 35 = 35\ and\ 1 * 35 = 1$$
$$\therefore Complement\ of\ 1\ is\ 35$$
$$5 \oplus 7 = 35\ and\ 5 * 7 = 1$$
$$\therefore Complement\ of\ 5\ is\ 7$$