Discrete Mathematics
Unit I Propositional and Predicate Calculus

What is proposition?
Solution:
A Proposition is a declarative sentence that is either true or false, but not both.
Eg: $2 > 1\ [True]$

$$1 + 7 = 9\ [False]$$

What is atomic statement? Give an example.
Solution:
Declarative sentences which cannot be further split into simpler sentences are called atomic statements.
Eg: Ram is a boy

What is compound statement? Give an example.
Solution:
Declarative sentences which can be further split into simpler sentences are called compound statement. Compound statements are constructed by combining the connectives ‘and’, ‘or’ , ‘but’ , etc.,

Write the truth table for negation?
Solution:
The negation of a statement is generally formed by introducing the word ‘not’ at a proper place in the statement.
Truth table for negation

P	¬P
T	F
F	T

Without using table prove the following
$P \wedge ((\neg P \wedge Q) \vee (\neg P \wedge \neg Q)) \Leftrightarrow R$
Solution:
$P \wedge ((\neg P \wedge Q) \vee (\neg P \wedge \neg Q)) \Leftrightarrow P \wedge (\neg P \wedge (Q \vee \neg Q))$
$\Leftrightarrow P \wedge (\neg P \wedge T)$
$\Leftrightarrow P \wedge \neg P$
$\Leftrightarrow F$
$\Leftrightarrow$ R

Express the statement “Good food is not cheap” in symbolic form.
Solution:
P: food is good.
Q: food is cheap
Symbolic form: $P \rightarrow \neg Q$

Obtain PDNF for $\neg P \vee Q$

Solution:

$$\neg P \vee Q \Leftrightarrow (\neg P \wedge (Q \vee \neg Q)) \vee ((P \vee \neg P) \wedge Q)$$

$$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q) \vee (\neg P \wedge Q)$$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$ which is PDNF

Write an equivalent formula for $P \wedge (Q \leftrightarrow R)$ which contains neither the biconditional nor the conditional.

Solution:

$$P \wedge (Q \leftrightarrow R) \Leftrightarrow P \wedge ((Q \rightarrow R) \wedge (R \rightarrow Q))$$

$$\Leftrightarrow P \wedge ((\neg Q \vee R) \wedge (\neg R \vee Q))$$

Write an equivalent formula for $P \rightarrow (Q \rightarrow R)$

Solution:

$P \rightarrow (Q \rightarrow R) \Leftrightarrow P \rightarrow (\neg Q \vee R) \Leftrightarrow \neg P \vee (\neg Q \vee R)$

Show that the propostion $(P \vee Q) \leftrightarrow (Q \vee P)$ is a tautology

Solution:

P	Q	$P \vee Q$	$Q \vee P$	$\neg(P \vee Q)$	$\neg(Q \vee P)$	$(P \vee Q) \leftrightarrow (Q \vee P)$
T	T	T	T	F	F	T
T	F	T	T	F	F	T
F	T	T	T	F	F	T
F	F	F	F	T	T	T

The last column contains only T.
Given proposition is tautology.

Show that $Q \vee (P \wedge \neg Q) \vee (\neg P \vee \neg Q)$ is a tautology.

Solution:

P	Q	$\neg Q$	$\neg P$	$P \wedge \neg Q$	$\neg P \vee \neg Q$	$Q \vee (P \wedge \neg Q) \vee (\neg P \vee \neg Q)$
T	T	F	T	F	T	T
T	F	T	T	T	T	T
F	T	F	F	F	F	T
F	F	T	F	F	T	T

The last column contains only T.
Given statement is tautology.

Using the truth table verify $(P \wedge Q) \wedge (\neg(PVQ))$ is contradiction.

Solution:

P	Q	P∧Q	P∨Q	¬ (P∨Q)	(P∧Q) ∧ (¬ (P∨Q))
T	T	T	T	F	F

T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

The last column contains only F
Given statement is contradiction.

Define contrapositive.
Solution:
If $P \rightarrow Q$ is an implication, then the converse of $P \rightarrow Q$ is the implication $Q \rightarrow P$ and the contrapositive of $P \rightarrow Q$ is the implication $\neg Q \rightarrow \neg P$

Give the converse and the contrapositive of the implication “If it is raining, then I get wet”
Solution: P: It is raining
Q: I get wet.
$Q \rightarrow P$ (Converse) If I get wet then it is raining
$\neg Q \rightarrow \neg P$(Contrapositive): If I do not get wet, then it is not raining.

Define the term “Logically equivalent”
Solution:
The propositions P and Q are called logically equivalent if $P \rightarrow Q$ is a tautology. It is denoted by $P \equiv Q$

Write the Statement “ The crop will be destroyed if there is a flood” in symbolic form
Solution: P: Crop will be destroyed
Q: There is a flood
Symbolic form: $Q \rightarrow P$

State and prove Duality principle theorem
Solution:
If $A\ and\ A^*$ be dual formulas and if $p_1, p_2, \dots p_n$ be simple variables that occur in $A\ and\ A^*$
ie) $A = A(p_1, p_2, \dots p_n)$ and $A^* = A^*(p_1, p_2, \dots p_n)$ then

$$\neg A(p_1, p_2, \dots p_n) \Leftrightarrow A^*(p_1, p_2, \dots p_n) and$$

$$A(\neg p_1, \neg p_2, \dots \neg p_n) \Leftrightarrow \neg A^*(p_1, p_2, \dots p_n)$$

That is the negation of a formula is equivalent to its dual in which every variable is replaced by its negation.

Define functionally complete sets of connectives.
Any set of connectives in which every formula can be expressed as another equivalent formula containing connectives from this set is called functionally complete set of connective.

Eg: The set of connectives $\{\wedge, \vee\}$

Prove that$\{\neg, \vee\}$ is a functionally complete set of connectives.
Solution:
It is enough to show that all formulas with other connectives, there exists a equivalent formula which contains $\neg$ and $\vee$ only.

Eg: i)$P \leftrightarrow Q \;\Leftrightarrow\; (P \rightarrow Q) \wedge (Q \rightarrow P)$

$\Leftrightarrow (\neg P \vee Q) \wedge (\neg Q \vee P)$

ii)$P \rightarrow Q \Leftrightarrow \neg P \vee Q$

iii)$P \wedge Q \;\Leftrightarrow\; \neg(\neg P \vee \neg Q)$

Hence $\{\neg, \vee\}$ is functionally complete set of connectives.

Show that $\{\wedge, \vee\}$ is not functionally complete.
Solution:
$\neg P$ cannot be expressed using the connectives $\{\wedge, \vee\}$. Since no such contribution of statement exist with $\{\wedge, \vee\}$ as input is T and the output is F.

Construct the truth table for NAND
Solution:

P	Q	$P \uparrow Q$
T	T	F
T	F	T
F	T	T
F	F	T

Obtain disjunctive normal forms of $P \wedge (P \rightarrow Q)$
Let $S \equiv P \wedge (P \rightarrow Q)$

$\Leftrightarrow P \wedge (\neg P \vee Q)$

$\Leftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$ which is DNF

Obtain a CNF for $(P \rightarrow (Q \wedge R)) \wedge (\neg P \rightarrow (\neg Q \wedge \neg R)$
Solution: $(P \rightarrow (Q \wedge R)) \wedge (\neg P \rightarrow (\neg Q \wedge \neg R))$

$\Leftrightarrow ((\neg P V (Q \wedge R)) \wedge (P \vee (\neg Q \wedge \neg R)$

$\Leftrightarrow ((\neg P \vee Q) \wedge (\neg P \vee R)) \wedge ((P \vee \neg Q) \wedge (P \vee \neg R))$ (Distributive law)

This is CNF, as it is a product of elementary sums.

Define Valid argument
If a conclusion is derived from a set of premised by using the accepted rules of reasoning, then such a process of derivation is called a deduction or a formal proof and the argument or conclusion is called a valid argument or valid conclusion.

Determine whether the conclusion c follows logically from the premised H_1, H_2 and H_3:
$H_1: P \rightarrow Q,\ H_2: P,\ H_3: Q$. Are given premises valid?

Solution:

P	Q	$P \rightarrow Q$	$(P \rightarrow Q) \land P$	$((P \rightarrow Q) \land P) \rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

The given premises are valid.

Write the rules for inference theory.

Rule P: A Premise may be introduced at any point in the derivation.

Rule T: A formula S may be introduced in a derivation if S is a tautologically implied by any one or more of the preceding formulas in the derivation.

Rule CP: If we can derive S from R and a set of premised, then we can derive $R \rightarrow S$ from the set of premises alone.

Demonstrate that R is valid inference from the premises $P \rightarrow Q, Q \rightarrow R \text{ and } P$

Solution:

$i)\ \ P \rightarrow Q \quad Rule\ P$

$ii)\ \ P \quad Rule\ P$

$iii)\ \ Q \quad Rule\ T\ From\ (i), (ii)\ and\ P, P \rightarrow Q \Rightarrow Q$

$iv)\ Q \rightarrow R \quad Rule\ P$

$V)\ \ R \quad Rule\ T\ \ From\ (iii), (iv)\ and\ Q, Q \rightarrow R \Rightarrow R$

Show that the following sets of premises are inconsistent.

$P \rightarrow Q\ , P \rightarrow R, Q \rightarrow \neg R, P$

Solution:

$i)\ \ P \rightarrow Q \quad Rule\ P$

$ii)\ \ Q \rightarrow \neg R \quad Rule\ P$

$iii)\ P \rightarrow \neg R \quad Rule\ T\ From\ (i), (ii)\ and\ P \rightarrow Q, Q \rightarrow R \Rightarrow\ \ P \rightarrow R$

$iv)\ \ P \quad Rule\ P$

$v\ )\ \ \neg R \quad Rule\ T\ From\ (iii), (iv)$

$vi)\ \ P \rightarrow R \quad Rule\ P$

$vii)\ R \quad Rule\ T\ \ From\ (iv), (vi), P, P \rightarrow R \Rightarrow\ \ R$

$viii) R \land \neg R \quad Rule\ T$

Given premises are inconsistent

If premises $P, Q\ and\ R$ are inconsistent, prove that $\neg R$ is a conclusion from $P\ and\ Q$.

Solution:

Given P, Q *and* R are inconsistent, $P \wedge Q \wedge R \Rightarrow F$ where F is contradiction

To prove: $P \wedge Q \Rightarrow \neg R$

Assume $P \wedge Q$ is true
If $\neg R$ is false $\Rightarrow$ R is true

Then only $P \wedge Q \wedge R$ is true which is contradiction.
$\neg R$ is true.
Hence $P \wedge Q \Rightarrow \neg R$

What is duality law of logical expression? Give the dual of $(P \vee F) \wedge (Q \vee T)$.
Solution:
In an expression, if we replace $\vee, \wedge, T, F$ respectively by $\wedge, \vee, F, T$. The resulting new formula is the dual of the given expression.
Dual of given formula is $(P \wedge T) \vee (Q \wedge F)$.

Define statement function of one variable. When it will become a statement?
Statement function is an expression containing symbols and an individual variable. It becomes a statement when the variable is replaced by particular value.

Use quantifiers to express the associative law for multiplication of real numbers.
$\forall x \; \forall y \; \forall z \; ((x.y).z = x.(y.z))$ where the universe of discourse for x, y and z is the set of real numbers.

Let the universe of discourse be $E = \{5,6,7\}$. Let $A = \{5,6\}$ *and* $B = \{6,7\}$.
Let $P(x)$: *x is in A*, $Q(x)$: *x is in B and* $R(x,y)$: $x + y < 12$.
Find the truth values of $(\exists x)(P(x) \rightarrow Q(x)) \rightarrow R(5,6)$
Solution:
$R(5,6)$ is true.
$P(5)$ is true and $Q(5)$ is false
$P(5) \rightarrow Q(5)$ is false
$P(6)$ is true and $Q(6)$ is true.
$P(6) \rightarrow Q(6)$ is true.
$P(7)$ is false and $Q(7)$ is true.
$P(7) \rightarrow Q(7)$ is false.
$(\exists x)(P(x) \rightarrow Q(x)$ is true.
Hence $(\exists x)(P(x) \rightarrow Q(x)) \rightarrow R(5,6)$ is true

Give an example in which $(\exists x)P(x) \rightarrow (\exists x)Q(x)$ is false.
Let the universe of discourse be $E = \{3,4,5\}$
Let $P(x)$: $x < 5$; $Q(x) : x > 7$
$P(3)$ is true.
$(\exists x)P(x)$ is true.
For any x *in* E, $Q(x)$ is false.
Hence $(\exists x)P(x) \rightarrow (\exists x)Q(x)$ is false

$P(6)$ is false and $Q(6)$ is false
$P(6) \rightarrow Q(6)$ is true.
$(\exists x)(P(x) \rightarrow Q(x))$ is true.

Find the truth value of $(x)(P \rightarrow Q(x)) \vee (x)R(x)$ where $P: 2 > 1, Q(x): x > 3$, $R(x): x > 4$ with the universe of discourse being $E = \{2,3,4\}$.
Solution:
P is true and $Q(4)$ is false, $P \rightarrow Q(4)$ is false
(x) (P→Q(x)) is false.
Since $R(2), R(3), R(4)$ are all false.
$(x)R(x)$ is false.
Hence $(x)(P \rightarrow Q(x)) \vee (\exists x)R(x)$ is false.

Define compound statement function.
A compound statement function is obtained by combining one or more simple statement functions by logical connectives.
Eg: $M(x) \wedge H(x), M(x) \rightarrow H(x), M(x) \wedge \neg H(x)$

Define Free and Bound variables.
When a quantifier is used on the variable x or when we assign a value to this variable, we say that this occurrence of the variable is bound. An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be free.
All the variables can be done using a combination of universal quantifiers, existential quantifiers and value assignments.
Eg: $(x)\ P(x, y)$
Here $P(x, y)$ is the scope of the quantifier and both occurrence of x are bound occurrences, while the occurrence of y is a free occurrence.

Let $P(x)$: x is a person
$F(x, y)$: x is the father of y
$M(x, y)$: x is the mother of y.
Write the predicate " x is the father of the mother of y".
We symbolize the predicate the name a person called z as the mother of y.
It is assumed that such a person z exists. We symbolize the predicate as

$$(\exists z)P(z) \wedge F(x, z) \wedge M(z, y)$$

Symbolize the expression "All the world loves a mother"
Let $P(x)$: x is a person
$M(x)$: x is a mother
$R(x, y)$: x loves y
The required expression is $(x)(P(x) \rightarrow (y)(P(y) \wedge M(y) \rightarrow R(x, y))$

Symbolize the statement " All men are giants:
$G(x)$: x is a gaint

$M(x)$: x is a man
Symbolically, $(x)\ (M(x) \rightarrow G(x))$

Symbolize: For every x, there exists a y such that $x^2 + y^2 \ \leq 100$.

$$(x)\ (\ \exists y)\ (x^2 + y^2 \ \leq 100)$$

Consider the statement " Give any positive integer, there is a greater positive integer". For all x, there exists a y such that y is greater than x. If $G(x, y)$ is " x is greater than y" then the given statement is $(x)(\ \exists y)\ G(y, x)$
If we do not impose the restriction on the universe of discourse and if we write $P(x)$ for " x is a positive integer" , then we can symbolize the given statement is

$$(x)(P(x) \rightarrow\ (\exists y)(P(y) \wedge G(y, x)))$$

Give the symbolic form of the statement " Every book with a blue cover is a mathematics book"
Let $B(x)$: x is every book with a blue cover
$M(x)$: x is mathematics book.
$(x)\ ((B(x) \rightarrow M(x))$

Write each of the following in symbolic form
i) All men are good.
ii) No men are good.

Solution:

i) All men are good.

$M(x)$: x is a man
$G(x)$: x is good
$(x)\ [M(x) \rightarrow G(x)]$

ii) No men are good
This can be written as, "For all x, if x is a man, then x is not good"

$$(x)[M(x) \rightarrow \neg G(x)]$$

Write each of the following in symbolic form
$i) Some\ men\ are\ good.$
$ii) Some\ men\ are\ not\ good.$
Solution:
i)Some men are good.
Let $M(x)$: x is a man
$G(x)$: x is good

$$(\exists x)(M(x) \wedge G(x))$$

ii) Some men are not good.

$$(\exists x)(M(x) \wedge \neg G(x))$$

Show that $(x)(H(x) \rightarrow M(x) \wedge H(s)) \Rightarrow M(s)$. Note that this problem is a symbolic translation of a well-known argument known as "Socrates argument" which is given by ,All men are mortal, Socrates is a man, Therefore Socrates is a mortal.
If we denote $H(x):$ x is a man, $M(x):$ x is a mortal, s : Socrates
We can put the argument in the above form.
$i)\ (x)(H(x) \rightarrow M(x)) \quad Rule\ P$
$ii)\ H(s) \rightarrow M(s) \quad Rule\ US\ From\ i)$
$iii)\ H(s) \quad Rule\ P$
$iv)\ M(s) \quad Rule\ T$

Verify the validity of the following argument.
All men are intelligent.
Krishna is a man.
Therefore Krishna is a intelligent.
Solution:
$P(x):$ x is man
$Q(x):$ x is intelligent
S : Krishna
We need to show $(x)(P(x) \rightarrow Q(x)) \wedge P(s) \Rightarrow Q(s)$
$i)\ (x)(P(x) \rightarrow Q(x)) \quad Rule\ P$
$ii) P(s) \rightarrow Q(s) \quad Rule\ US$
$iii)\ P(s) \quad Rule\ P$
$iv) Q(s) \quad Rule\ T\ From\ ii)\ \&\ iii)$.

Define universe of discourse.
The variables which are quantified stand for only those objects which are members of a particular set or class. Such a restricted class is called the universe of discourse or the domain of individuals or simply the universe.

Consider the statement " Given any positive integer, there is a greater positive integer" . Symbolize this statement with and without using the set of positive integers as the universe of discourse.
For all x, there exists a y such that y is greater than x. If $G(x, y)$ is " x is greater than y", then the given statement is $(x)(\exists\ y)\ G(y, x)$.
If we do not impose the restriction on the universe of discourse and if we writer $P(x)$ for " x is a positive integer", then we can symbolize the given statement is

$$(x)(P(x) \rightarrow (\exists y)(P(y) \wedge G(y, x)))$$

Define Universal quantifiers
The universal quantification of $P(x)$ is the proposition. "$P(x)$ is true for all values of x in the universe of discourse"
The notation $(x)P(x)$ denotes the universal quantification of $P(x)$. Here (x) is called the universal quantifier.

Define existential quantifier.
The existential quantification of $P(x)$ is the proposition. "There exists an element x in the universe of discourse such that P(x) is true"
We use the notation $(\exists x)(P(x))$ for the existential quantification of $P(x)$. Here $\exists$ is called the existential quantifier.

Write the universal specification in quantifiers.
From $(x)\ A(x)$ one can conclude $A(y)$. If $(x)\ A(x)$ is true for every element x in the universe, then $A(y)$ is true.

$$(x)A(x) \Rightarrow A(y)$$

Define Existential specification in quantifiers
From $(\exists x)\ A(x)$ one can conclude $A(y)$. If $(\exists x)\ A(x)$ is true for some element x in the universe, then $A(y)$ is true.

$$(\exists x)A(x) \Rightarrow A(y)$$

Define Existential Generalization.
From $A(x)$ one can conclude $(\exists y)\ A(y)$. If $A(x)$ is true for some element x in the universe, then $(\exists y)\ A(y)$ is true.

$$A(x) \ \Rightarrow (\exists y)A(y)$$

Define Universal Generalization
From $A(x)$ one can conclude $(y)A(y)$. If $A(x)$ is true for every element x in the universe, then $(y)\ A(y)$ is true.

$$A(x) \ \Rightarrow (y)A(y)$$

Show that $\neg P(a,b)$ follows logically from $(x)(y)(P(x,y) \rightarrow w(x,y))$ and $\neg w(a,b)$
Solution:
i) $(x)(y)(P(x,y) \rightarrow w(x,y))$ Given premise
ii) $(y)\ P(a,y) \rightarrow w(a,y)$ US
iii) $P(a,b) \rightarrow w(a,b)$ US
iv) $\neg w(a,b)$ Rule P
v) $\neg P(a,b)$ (iii),(iv), Modus tollens

If the universe of discourse is finite, then show that $\neg[(\exists x)P(x)] \Leftrightarrow (x)[\neg P(x)]$.

Solution:
Let the universe of discourse be $U = \{x_1, x_2, \dots, x_n\}$ be finite.
By using DeMorgan's Law of propositional calculus, we have

$$\begin{aligned}\neg[(\exists x)P(x)] &\Leftrightarrow \neg[P(x_1) \vee P(x_2) \dots \vee P(x_n)] \\ &\Leftrightarrow \neg P(x_1) \wedge \neg P(x_2) \wedge \dots \wedge \neg P(x_n) \\ &\Leftrightarrow (x)[\neg P(x)]\end{aligned}$$

Part B

Without using truth table, show that $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$

Solution:

$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$

$\Leftrightarrow \big(\neg P \wedge (\neg Q \wedge R)\big) \vee \big((Q \vee P) \wedge R\big)$ $[Distributive\ law]$

$\Leftrightarrow \big((\neg P \wedge \neg Q) \wedge R\big) \vee \big((Q \vee P) \wedge R\big)$ $[Associative\ law]$

$\Leftrightarrow (\neg (P \vee Q) \wedge R) \vee \big((Q \vee P) \wedge R\big)$ $[De\ morgan's\ law]$

$\Leftrightarrow (\neg (P \vee Q) \wedge R) \vee \big((P \vee Q) \wedge R\big)$ $[commutative\ law]$

$\Leftrightarrow \big(\neg (P \vee Q) \vee (P \vee Q)\big) \wedge R$ $[Distributive\ law]$

$\Leftrightarrow T \wedge R$ $\left[\neg P \wedge P \Rightarrow T\right]$

$\Leftrightarrow R$ $\left[T \wedge P \Rightarrow P\right]$

Without using truth table, show that $(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Leftrightarrow (\neg P \wedge Q)$

Solution:

$(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q))$

$\Leftrightarrow (P \vee Q) \wedge ((\neg P \wedge \neg P) \wedge Q)$ $[Associative\ law]$

$\Leftrightarrow (P \vee Q) \wedge (\neg P \wedge Q)$ $[Idempotent\ law]$

$\Leftrightarrow (P \wedge (\neg P \wedge Q)) \vee (Q \wedge (\neg P \wedge Q))$ $[Distributive\ law]$

$\Leftrightarrow ((P \wedge \neg P) \wedge Q) \vee ((Q \wedge \neg P) \wedge Q)$ $[Associative\ law]$

$\Leftrightarrow \big((P \wedge \neg P) \vee (Q \wedge \neg P)\big) \wedge Q$ $[Distributive\ law]$

$\Leftrightarrow \big(F \vee (Q \wedge \neg P)\big) \wedge Q$ $\left[P \wedge \neg P \Rightarrow F\right]$

$\Leftrightarrow (Q \wedge \neg P) \wedge Q$ $\left[F \vee P \Rightarrow P\right]$

$\Leftrightarrow (\neg P \wedge Q) \wedge Q$ $[Commutative\ law]$

$\Leftrightarrow \neg P \wedge (Q \wedge Q)$ $[Associative\ law]$

$\Leftrightarrow \neg P \wedge Q$ $[Idempotent\ law]$

Without using truth table obtain disjunctive normal forms of

$\neg (P \vee Q) \leftrightarrow (P \wedge Q)$

Solution:

$$\neg (P \vee Q) \leftrightarrow (P \wedge Q)$$

$\equiv \Big(\neg (P \vee Q) \rightarrow (P \wedge Q)\Big) \wedge \Big((P \wedge Q) \rightarrow \neg (P \vee Q)\Big)$

$\equiv (\neg\neg (P \vee Q) \vee (P \wedge Q)) \wedge (\neg (P \wedge Q) \vee \neg (P \vee Q))$

$\equiv ((P \vee Q) \vee (P \wedge Q)) \wedge ((\neg P \vee \neg Q) \vee (\neg P \wedge \neg Q))$

$\equiv \big(((P \vee Q) \vee P) \wedge ((P \vee Q) \vee Q)\big) \wedge ((\neg P \vee \neg Q \vee \neg P) \wedge (\neg P \vee \neg Q \vee \neg Q))$

$\equiv ((P \vee Q) \wedge (P \vee Q)) \wedge ((\neg P \vee \neg Q) \wedge (\neg P \vee \neg Q))$

$\equiv (P \vee Q) \wedge (\neg P \vee \neg Q)$

$\equiv ((P \vee Q)\wedge\neg P) \vee((P \vee Q)\wedge\neg Q)$
$\equiv (P \wedge\neg P)\vee (Q\wedge\neg P) \vee(P\wedge\neg Q) \vee (Q\wedge\neg Q)$
$\equiv F\vee (Q\wedge\neg P) \vee(P\wedge\neg Q) \vee F$
$(Q\wedge\neg P) \vee(P\wedge\neg Q)$

Without using truth table obtain conjunctive normal forms of
$P \rightarrow ((P \rightarrow Q) \wedge \neg(\neg Q\vee\neg P))$
Solution:
$P \rightarrow ((P \rightarrow Q) \wedge \neg(\neg Q\vee\neg P))$
$\equiv P \rightarrow ((\neg P\vee Q) \wedge \neg(\neg Q\vee\neg P))$
$\equiv P \rightarrow ((\neg P\vee Q) \wedge (Q\wedge P))$
$\equiv P \rightarrow (\neg P \wedge (Q\wedge P))\vee(Q \wedge (Q\wedge P))$
$\equiv P \rightarrow (\neg P \wedge (P\wedge Q))\vee(Q \wedge (Q\wedge P))$
$\equiv P \rightarrow ((\neg P \wedge P)\wedge Q)\vee((Q \wedge Q)\wedge P)$
$\equiv P \rightarrow F\vee(Q\wedge P)$
$\equiv P \rightarrow (Q\wedge P)$
$\equiv \neg P\vee (Q\wedge P)$
$\equiv (\neg P\vee Q)\wedge(\neg P\vee P)$
$\equiv (\neg P\vee Q)\wedge T$
$\equiv \neg P\vee Q$ which is CNF
Without constructing the truth table obtain the product of sums canonical form of the formula $(\neg P \rightarrow R) \wedge (Q \leftrightarrow R)$

Solution:
$(\neg P \rightarrow R) \wedge (Q \leftrightarrow R)$
$\equiv (\neg\neg P\vee R) \wedge (Q \rightarrow R)\wedge (R \rightarrow Q)$
$\equiv (P\vee R) \wedge (Q \rightarrow R)\wedge (R \rightarrow Q)$
$\equiv (P\vee R) \wedge (\neg Q \vee R)\wedge (\neg R \vee Q)$
$\equiv (P\vee R\vee(Q\wedge\neg Q)) \wedge ((P\wedge\neg P)\vee\neg Q \vee R)\wedge ((P\wedge\neg P)\vee\neg R \vee Q)$
$\equiv (P\vee R\vee Q) \wedge (P\vee R\vee\neg Q)\wedge(P\vee\neg Q \vee R)\wedge(\neg P\vee\neg Q \vee R)$
$\wedge (P\vee\neg R \vee Q)\wedge (\neg P\vee\neg R \vee Q)$
$\equiv (P\vee R\vee Q) \wedge (P\vee\neg Q \vee R)\wedge(\neg P\vee\neg Q \vee R)$
$\wedge (P\vee\neg R \vee Q)\wedge (\neg P\vee\neg R \vee Q)$ writing the repeating terms only once
which is the PCNF of $(\neg P \rightarrow R) \wedge (Q \leftrightarrow R)$

Without using truth table obtain the product of sums canonical form of
$(P\wedge Q\wedge R)\vee(\neg P\wedge Q\wedge R)\vee(\neg P\wedge\neg Q\wedge\neg R)$
Solution:
$Let\ S \equiv (P\wedge Q\wedge R)\vee(\neg P\wedge Q\wedge R)\vee(\neg P\wedge\neg Q\wedge\neg R)$ which is PDNF
$\neg S\ represents\ the\ missing\ terms\ in\ PDNF$
$\neg S \equiv (P\wedge\neg Q\wedge R)\vee(P\wedge Q\wedge\neg R)\vee(\neg P\wedge\neg Q\wedge R)\vee(\neg P\wedge Q\wedge\neg R)$
$\vee(P\wedge\neg Q\wedge\neg R)$
$\neg\neg S \equiv \neg(P\wedge\neg Q\wedge R)\wedge\neg(P\wedge Q\wedge\neg R)\wedge\neg(\neg P\wedge\neg Q\wedge R)\wedge\neg(\neg P\wedge Q\wedge\neg R)$
$\wedge\neg(P\wedge\neg Q\wedge\neg R)$

$S \equiv (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee R)$
$\wedge (\neg P \vee Q \vee R)$ which is PCNF

Without constructing the truth table obtain PDNF of $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$. Also find PCNF.

Solution:

$Let\ S \equiv (P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

$\equiv ((P \wedge Q) \wedge (R \vee \neg R)) \vee (\neg P \wedge (Q \vee \neg Q) \wedge R) \vee ((P \vee \neg P) \wedge (Q \wedge R))$

$\equiv (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee \left(((\neg P \wedge Q) \vee (\neg P \wedge \neg Q)) \wedge R \right)$
$\vee (P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge R)$

$\equiv (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$
$\vee (P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge R)$

$\equiv (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$

which is PDNF

$\neg S\ represents\ the\ missing\ terms\ in\ PDNF$

$\neg S \equiv (P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R)$

$\neg\neg S \equiv \neg(P \wedge \neg Q \wedge R) \wedge \neg(P \wedge \neg Q \wedge \neg R) \wedge \neg(\neg P \wedge Q \wedge \neg R) \wedge \neg(\neg P \wedge \neg Q \wedge \neg R)$

$S \equiv (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee Q \vee R)$

which is PCNF

Without constructing the truth table show that

$$\neg(P \wedge Q) \to (\neg P \vee (\neg P \vee Q)) \Leftrightarrow (\neg P \vee Q)$$

Solution:

$\neg(P \wedge Q) \to (\neg P \vee (\neg P \vee Q))$

$\Leftrightarrow \neg(P \wedge Q) \to ((\neg P \vee \neg P) \vee Q)$

$\Leftrightarrow \neg(P \wedge Q) \to (\neg P \vee Q)$

$\Leftrightarrow \neg\neg(P \wedge Q) \vee (\neg P \vee Q)$

$\Leftrightarrow (P \wedge Q) \vee (\neg P \vee Q)$

$\Leftrightarrow (P \vee (\neg P \vee Q)) \wedge (Q \vee (\neg P \vee Q))$

$\Leftrightarrow ((P \vee \neg P) \vee Q) \wedge (Q \vee (\neg P \vee Q))$

$\Leftrightarrow (T \vee Q) \wedge (Q \vee (\neg P \vee Q))$

$\Leftrightarrow T \wedge (Q \vee (\neg P \vee Q))$

$\Leftrightarrow Q \vee (\neg P \vee Q)$

$\Leftrightarrow (\neg P \vee Q) \vee Q$

$\Leftrightarrow \neg P \vee (Q \vee Q)$

$\Leftrightarrow \neg P \vee Q$

Without constructing the truth table show that $(P \wedge Q) \to (P \vee Q)$ is a tautology.

Solution:

$(P \wedge Q) \to (P \vee Q)$

$\Rightarrow \neg(P \wedge Q) \vee (P \vee Q)$

$\Rightarrow (\neg P \lor \neg Q) \lor (P \lor Q)$

$\Rightarrow \neg P \lor (\neg Q \lor P) \lor Q$

$\Rightarrow \neg P \lor (P \lor \neg Q) \lor Q$

$\Rightarrow (\neg P \lor P) \lor (\neg Q \lor Q)$

$\Rightarrow T \lor T$

$\Rightarrow T$

$(P \land Q) \to (P \lor Q)$ is a tautology

Without constructing the truth table obtain the PDNF of $\neg P \lor Q$. Also find PCNF.
Solution:
Let $S \equiv \neg P \lor Q$
$\equiv \left(\neg P \land (Q \lor \neg Q)\right) \lor \left((P \lor \neg P) \land Q\right)$
$\equiv (\neg P \land Q) \lor (\neg P \land \neg Q) \lor (P \land Q) \lor (\neg P \land Q)$
$\equiv (\neg P \land Q) \lor (\neg P \land \neg Q) \lor (P \land Q)$ *is a* PDNF
$\neg S \equiv P \land \neg Q$
$\neg\neg S \equiv \neg (P \land \neg Q)$
$S \equiv \neg P \lor Q$ is a PCNF

Without constructing the truth table show that $R \lor S$ from the following premises, $C \lor D, (C \lor D) \to \neg H, \neg H \to (A \land \neg B)$ *and* $(A \land \neg B) \to (R \lor S)$.
Solution:

$i)\ C \lor D$	*Rule P*
$ii)\ (C \lor D) \to \neg H$	*Rule P*
$iii)\ \neg H \to (A \land \neg B)$	*Rule P*
$iv)\ (C \lor D) \to (A \land \neg B)$	*Rule T, ii , iii and hypothetical syllogism*
$v)\ (A \land \neg B) \to (R \lor S)$	*Rule P*
$vi)\ (C \lor D) \to (R \lor S)$	*Rule T, iv , v and hypothetical syllogism*
$vii)\ R \lor S$	*Rule T, i, vi and modus ponens*

Without constructing the truth table show that $R \land (P \lor Q)$ is a valid conclusion from the premises $P \lor Q, Q \to R, P \to M$ *and* $\neg M$.

$i)\ P \lor Q$	*Rule P*
$ii)\ P \to M$	*Rule P*
$iii)\ \neg M$	*Rule P*
$iv)\ \neg P$	*Rule T, ii, iii and modus tollens*
$v)\ Q$	*Rule T, i, iv and disjunctive syllogism*
$vi)\ Q \to R$	*Rule P*
$vii)\ R$	*Rule T, v, vi and modus ponens*
$viii)\ R \land (P \lor Q)$	*Rule T, vii, i and conjuction*

Show that the following premises are inconsistent.
If Jack misses many classes through illness, then he fails high school.

If Jack fails high school, then he is uneducated
If Jack reads a lot of books, then he is not uneducated.
Jack misses many classes through illness and reads a lot of books.
Solution:

Let C represents Jack misses many classes through illness
Let F represents Jack fails high school
Let E represents Jack is uneducated
Let B represents Jack reads lot of books

The symbolic representation of the problem is

$C \rightarrow F, F \rightarrow E, B \rightarrow \neg E, C \wedge B$ are inconsistent.

$i)\ C \wedge B \qquad Rule\ \ P$
$ii)\ C \qquad Rule\ T, i, and\ Simplification$
$iii)\ B \qquad Rule\ T, i\ and\ Simplification$
$iv)\ C \rightarrow F \qquad Rule\ P$
$v)\ F \rightarrow E \qquad Rule\ P$
$vi)\ C \rightarrow E \qquad Rule\ T, iv, v\ and\ hypothetical\ syllogism$
$vii)\ B \rightarrow \neg E \quad Rule\ p$
$viii)\ \neg E \qquad Rule\ T, iii, vii\ and\ modus\ ponens$
$ix)\ E \qquad Rule\ T, ii, vi\ and\ modus\ ponens$
$x)\ E \wedge \neg E \qquad Rule\ T, viii, xi\ and\ conjuction$
$xi)\ F \qquad Rule\ T, x\ and\ negation\ law$

The set of given premises are inconsistent.

Show that the following set of premises is inconsistent.
If the contract is valid, then John is liable for penalty. If John is liable for penalty, he will go bankrupt. If the bank will loan him money, he will not go bankrupt. As a matter of fact, the contract is valid and bank will loan him money.
Solution:

Let C represents the contract is valid
Let P represents John is liable for penalty
Let B represents John will go bankrupt

Let M represents bank will loan him money
The symbolic representation of the problem is
$C \rightarrow P, P \rightarrow B, M \rightarrow \neg B, C \wedge M$ are inconsistent.

$i)\ C \wedge M \qquad Rule\ \ P$
$ii)\ C \qquad Rule\ T, i, and\ Simplification$
$iii)\ M \qquad Rule\ T, i, and\ Simplification$
$iv)\ C \rightarrow P \qquad Rule\ P$
$v)\ P \rightarrow B \qquad Rule\ P$
$vi)\ C \rightarrow B \qquad Rule\ T, iv, v\ and\ hypothetical\ syllogism$

$vii)\ M \rightarrow \neg B\ Rule\ p$

$viii)\ \neg B \quad Rule\ T, iii, vii\ and\ modus\ ponens$
$ix)\ B \quad Rule\ T, ii, vi\ and\ modus\ ponens$
$x)\ B \wedge \neg B \quad Rule\ T, viii, xi\ and\ conjuction$
$xi)\ F \quad Rule\ T, x\ and\ negation\ law$

The set of given premises are inconsistent.

By Indirect proof, Show that $P \rightarrow Q, Q \rightarrow R, P \vee R \Rightarrow R$.

Solution:
$Let\ us\ assume\ that\ \neg R\ be\ the\ additional\ premises\ and$
$prove\ a\ contradiction$
$i)\ \neg R \quad Rule\ Additional\ P$
$ii)\ Q \rightarrow R \quad Rule\ P$
$iii)\ \neg Q \quad Rule\ T, i, ii\ and\ modus\ tollens$
$iv)\ P \rightarrow Q \quad Rule\ P$
$v)\ \neg P \quad Rule\ T, iii, iv\ and\ modus\ tollens$
$vi)\ \neg P \wedge \neg R \quad Rule\ T, iii, v\ and\ conjuction$
$vii)\ \neg(P \vee R) \quad Rule\ T, vi\ and\ Demorgans\ law$
$viii)\ P \vee R \quad Rule\ P$
$ix)\ \neg(P \vee R) \wedge (P \vee R) \quad Rule\ T, vii, viii\ and\ conjuction$
$x)\ F \quad Rule\ T, ix\ and\ \ Negation$

Without using truth tables, show that $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is tautology.
Solution:
$Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$

$\Rightarrow \big((Q \vee P) \wedge (Q \vee \neg Q)\big) \vee (\neg P \wedge \neg Q)$ $[Distributive\ law]$

$\Rightarrow \big((Q \vee P) \wedge T\big) \vee (\neg P \wedge \neg Q)$ $[Q \vee \neg Q = T]$

$\Rightarrow (Q \vee P) \vee (\neg P \wedge \neg Q)$ $[Q \wedge T = Q]$

$\Rightarrow \big((Q \vee P) \vee \neg P\big) \wedge \big((Q \vee P) \vee \neg Q\big)$ $[Distributive\ law]$

$\Rightarrow \big(Q \vee (P \vee \neg P)\big) \wedge \big((P \vee Q) \vee \neg Q\big)$ $[Associative\ \&\ commutative\ law]$

$\Rightarrow \big(Q \vee (P \vee \neg P)\big) \wedge \big(P \vee (Q \vee \neg Q)\big)$ $[Associative\ law]$

$\Rightarrow (Q \vee T) \wedge (P \vee T)$ $[Q \vee \neg Q = T]$

$\Rightarrow T \wedge T$ $[Q \vee T = T]$

$\Rightarrow T$

$Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is tautology

Without constructing the truth table show that S is valid inference from the premises $P \rightarrow \neg Q, Q \vee R, \neg S \rightarrow P\ and\ \neg R.$
Solution:
$i)\ Q \vee R \quad Rule\ P$
$ii)\ \neg R \quad Rule\ P$

$iii)\ Q$ $\quad Rule\ T, i, ii\ and\ dusjunction\ syllogism$
$iv)\ P \rightarrow \neg Q$ $\quad Rule\ P$
$v)\ \neg P$ $\quad Rule\ T, iii, iv\ and\ Modus\ tollens$
$vi)\ \neg S \rightarrow P$ $\quad Rule\ P$
$vii)\ \neg\neg S$ $\quad Rule\ T, v, vi\ and\ Modus\ tollens$
$viii)\ S$ $\quad Rule\ T, vii\ and\ Negation$

Show that $(x)((P(x) \rightarrow Q(x))\ \wedge (Q(x) \rightarrow R(x)) \Rightarrow\ (x)(P(x) \rightarrow R(x))$

Solution:

1. $(x)((P(x) \rightarrow Q(x))\ \wedge (Q(x) \rightarrow R(x))$ $\quad Rule\ P$
2. $(P(a) \rightarrow Q(a))\ \wedge (Q(a) \rightarrow R(a))$ $\quad Rule\ US$,1
3. $P(a) \rightarrow Q(a)$ $\quad Rule\ T$,2
4. $Q(a) \rightarrow R(a)$ $\quad Rule\ T$,2
5. $P(a) \rightarrow R(a)$ $\quad Simplification$, 3,4
6. $(x)(P(x) \rightarrow R(x))$ $\quad Rule\ UG$,5

Show that $(\exists x)\ M(x)$ follows logically from the premises.

$(x)(H(x) \rightarrow M(x))\ and\ (\exists x)H(x)$

Solution:

1. $(x)\big(H(x) \rightarrow M(x)\big)$ $\quad Rule\ P$
2. $H(a) \rightarrow M(a)$ $\quad Rule\ US$,1
3. $(\exists x)H(x)$ $\quad Rule\ P$
4. $H(a)$ $\quad Rule\ ES$, 3
5. $M(a)$ $\quad Rule\ Modus\ ponens$,2,4
6. $(\exists x)M(x)$ $\quad Rule\ EG$,5

Prove that $(\exists x)\big(P(x) \wedge Q(x)\big) \Rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$

Solution:

1. $(\exists x)(P(x) \wedge Q(x))$ $\quad Rule\ P$
2. $P(a) \wedge Q(a)$ $\quad Rule\ ES$,1
3. $P(a)$ $\quad Rule\ T$, 2
4. $(\exists x)P(x)$ $\quad Rule\ EG$, 3
5. $Q(a)$ $\quad Rule\ T$, 2
6. $(\exists x)Q(x)$ $\quad Rule\ EG$,5
7. $(\exists x)P(x) \wedge (\exists x)Q(x)$ $\quad Rule\ Conjuction$,4,6

Show that from $(\exists x)\big(F(x) \wedge S(x)\big) \rightarrow\ (y)\big(M(y) \rightarrow w(y)\big)$,

$(\exists y)(M(y) \wedge \neg w(y)) \Rightarrow\ (x)F(x) \rightarrow\ \neg S(x)$

Solution:

1. $(\exists y)(M(y) \wedge \neg w(y))$ $\quad Rule\ P$
2. $M(a) \wedge \neg w(a)$ $\quad Rule\ ES$,1
3. $\neg(\neg M(a) \vee w(a))$ $\quad 2, De\ Morgan's\ law$
4. $\neg\big(M(a) \rightarrow w(a)\big)$ $\quad 3, disjunction\ as\ conditional$
5. $(\exists x)\big(F(x) \wedge S(x)\big) \rightarrow\ (y)\big(M(y) \rightarrow w(y)\big)$ $\quad Rule\ P$

6. $(F(b) \wedge S(b)) \rightarrow (M(a) \rightarrow w(a))$ — *Rule ES*, 5
7. $\neg(F(b) \wedge S(b))$ — *Modus tollens*, 4,6
8. $\neg F(b) \vee \neg S(b)$ — 7, *De Morgan's law*
9. $F(b) \rightarrow \neg S(b)$ — 8, *disjunction as conditional*
10. $(x)F(x) \rightarrow \neg S(x)$ — 9, *UG*

Show that $(x)\,(P(x) \vee Q(x)) \Rightarrow (x)P(x) \vee (\exists x)Q(x)$

Solution:

Let us prove this by indirect method

Let us assume that $\neg((x)P(x) \vee (\exists x)Q(x))$ as additional premise

1. $\neg((x)P(x) \vee (\exists x)Q(x))$ — *Additional premise*
2. $\neg(x)\,P(x) \wedge \neg(\exists x)Q(x)$ — 1, *De Morgan's law*
3. $\neg(x)\,P(x)$ — *Rule T*, 2
4. $(\exists x)\neg P(x)$ — 3, *De Morgan's law*
5. $\neg P(a)$ — *Rule ES*, 4
6. $\neg(\exists x)Q(x)$ — *Rule T*, 2
7. $(x)\neg Q(x)$ — 6, *De Morgan's law*
8. $\neg Q(a)$ — *Rule US*, 7
9. $\neg P(a) \wedge \neg Q(a)$ — 5,8, *conjunction*
10. $\neg(P(a) \vee Q(a))$ — 9, *De Morgan's law*
11. $(x)\,(P(x) \vee (Q(x))$ — *Rule P*
12. $P(a) \vee Q(a)$ — *Rule US*, 11
13. $\neg(P(a) \vee Q(a)) \wedge (P(a) \vee Q(a))$ — 11,12, *conjunction*
14. F — *Rule T*, 13

There is mistake in the following derivation. Find it. Is the conclusion valid?. If so, obtain a correct derivation.

1. $(x)(P(x) \rightarrow Q(x))$ — *Rule P*
2. $P(y) \rightarrow Q(y)$ — *US*
3. $(\exists x)P(x)$ — *Rule P*
4. $P(y)$ — *ES*
5. $Q(y)$ — *Rule T*, 2,4
6. $(\exists x)Q(x)$ — *EG*

Solution:

1. $(x)(P(x) \rightarrow Q(x))$ — *Rule P*
2. $P(a) \rightarrow Q(a)$ — *Rule US*, 2
3. $(\exists y)P(y)$ — *Rule P*
4. $P(a)$ — *Rule ES*, 3
5. $Q(a)$ — *T*, 2,4, *and modus ponens*
6. $(\exists z)Q(z)$ — *Rule EG*, 5

Therefore $(\exists z)Q(z)$ is validly derivable from the premises $(x)(P(x) \rightarrow Q(x)), (\exists y)P(y)$

Obtain the following implication by indirect method.

$(x)(P(x) \to Q(x)), (x)(R(x) \to \neg Q(x)) \Rightarrow (x)(R(x) \to \neg P(x))$

Solution:

Contrapositive method:

Let us assume that $\neg(x)(R(x) \to \neg P(x))$ as additional premise.

1. $\neg(x)\big(R(x) \to \neg P(x)\big)$ — *Rule additional P*
2. $(\exists x)\neg\big(R(x) \to \neg P(x)\big)$ — Demorgan's law ,1
3. $\neg\big(R(a) \to \neg P(a)\big)$ — $2, ES$
4. $\neg\big(\neg R(a) \vee \neg P(a)\big)$ — $T, 2,$ *and equivalence*
5. $R(a) \wedge P(a)$ — Demorgan's law ,4
6. $R(a)$ — $T, 5$
7. $P(a)$ — $T, 5$
8. $(x)(R(x) \to \neg Q(x))$ — *Rule P*
9. $R(a) \to \neg Q(a)$ — *Rule US* ,3
10. $\neg Q(a)$ — $T, 6,9,$ *and modus ponens*
11. $P(a) \wedge \neg Q(a)$ — $T, 7,10,$ *and* conjuction
12. $\neg\big(\neg P(a) \vee Q(a)\big)$ — $T, 6,$ Demorgan's law
13. $\neg(P(a) \to Q(a))$ — $T, 12,$ *and equivalence*
14. $(x)(P(x) \to Q(x))$ — *Rule P*
15. $P(a) \to Q(a)$ — *Rule US* ,14
16. $\big(\neg(P(a) \to Q(a))\big) \wedge P(a) \to Q(a)$ — $T, 13,15$ *and conjuction*
17. F — $T, 16$ *and negation law*

Is the following conclusion validly derivable from the premises given?

$$(x)\big(P(x) \to Q(x)\big), (\exists y)P(y) \Rightarrow (\exists z)Q(z).$$

Solution:

1. $(x)(P(x) \to Q(x))$ — *Rule P*
2. $P(a) \to Q(a)$ — *Rule US* ,2
3. $(\exists y)P(y)$ — *Rule P*
4. $P(a)$ — *Rule ES* ,3
5. $Q(a)$ — $T, 2,4,$ *and modus ponens*
6. $(\exists z)Q(z)$ — *Rule EG* ,5

Therefore $(\exists z)Q(z)$ is validly derivable from the premises

$(x)\big(P(x) \to Q(x)\big), (\exists y)P(y)$

Use indirect method of proof show that

$$(\exists x)(A(x) \vee B(x)) \Leftrightarrow (\exists x)A(x) \vee (\exists x)B(x)$$

Solution:

Let us assume that $\neg((\exists x)A(x) \vee (\exists x)B(x))$ as additional premise

1. $\neg((\exists x)A(x) \vee (\exists x)B(x))$ — *Additional premise*
2. $\neg(\exists x)A(x) \wedge \neg(\exists x)B(x)$ — $1,$ *De Morgan's law*
3. $(x)\neg A(x) \wedge (x)\neg B(x)$ — $2,$ *De Morgan's law*
4. $\neg A(a) \wedge \neg B(a)$ — *Rule US*, 3
5. $\neg(A(a) \vee B(a))$ — $4,$ *De Morgan's law*

6. $(\exists x)(A(x) \vee B(x))$ — $Rule\ P$
7. $A(a) \vee B(a)$ — $Rule\ ES, 6$
8. $\neg\, (A(a) \vee B(a)) \wedge (A(a) \vee B(a))$ — $5,7, conjunction$
9. F — $Rule\ T, 8\ and\ negation\ law$

Obtain the following implication.

$$(x)(P(x) \rightarrow Q(x)), (x)(R(x) \rightarrow \neg Q(x)) \Rightarrow (x)(R(x) \rightarrow \neg P(x))$$

Solution:

1. $(x)(P(x) \rightarrow Q(x))$ — $Rule\ P$
2. $P(a) \rightarrow Q(a)$ — $Rule\ US\ ,1$
3. $(x)(R(x) \rightarrow \neg Q(x))$ — $Rule\ P$
4. $R(a) \rightarrow \neg Q(a)$ — $Rule\ US\ ,3$
5. $Q(a) \rightarrow \neg R(a)$ — $T, 4, and\ equivalence$
6. $P(a) \rightarrow \neg R(a)$ — $T, 2{,}5, and$ hypothetical syllogism
7. $R(a) \rightarrow \neg P(a)$ — $T, 6, and\ equivalence$
8. $(x)(R(x) \rightarrow \neg P(x))$ — $7, UG$

Prove that $(\exists x)(P(x) \wedge S(x)), (x)\ (P(x) \rightarrow R(x)) \Rightarrow (\exists x)(R(x) \wedge S(x))$

Solution:

1. $(x)(P(x) \rightarrow R(x))$ — $Rule\ P$
2. $P(a) \rightarrow R(a)$ — $Rule\ US\ ,2$
3. $(\exists x)(P(x) \wedge S(x))$ — $Rule\ P$
4. $P(a) \wedge S(a)$ — $Rule\ ES\ ,3$
5. $P(a)$ — $T, 4, and\ conjuction$
6. $S(a)$ — $T, 4, and\ conjuction$
7. $R(a)$ — $T, 2{,}5, and\ modus\ ponens$
8. $R(a) \wedge S(a)$ — $T, 6{,}7\ and\ conjuction$
9. $(\exists x)\big(R(x) \wedge S(x)\big)$ — $EG, 8$

By indirect method prove that $(x)\ (P(x) \rightarrow Q(x)), (\exists x)P(x) \Rightarrow (\exists x)Q(x)$

Solution:

Let us assume that $\neg(\exists x)Q(x)$ as additional premise

1. $\neg(\exists x)Q(x)$ — $Additional\ premise$
2. $(x)\neg Q(x)$ — $1, De\ Morgan's\ law$
3. $\neg Q(a)$ — $Rule\ US, 2$
4. $(\exists x)P(x)$ — $Rule\ P$
5. $P(a)$ — $Rule\ ES, 4$
6. $P(a) \wedge \neg Q(a)$ — $5{,}3\ and\ conjuction$
7. $\neg\big(\neg P(a) \vee Q(a)\big)$ — $6, De\ Morgan's\ law$
8. $\neg\big(P(a) \rightarrow Q(a)\big)$ — $T, 7, Equivalence$
9. $(x)\big(P(x) \rightarrow Q(x)\big)$ — $Rule\ P$
10. $P(a) \rightarrow Q(a)$ — $Rule\ US, 9$

11. $\neg\big(P(a) \to Q(a)\big) \land P(a) \to Q(a)$ 8,10 *and conjuction*
12. F *Rule* T, 11 *and negation law*

$(x)(H(x) \to A(x)) \Rightarrow (x)\,((\exists y)(H(y) \land N(x,y)) \to (\exists y)(A(y) \land N(x,y)))$

Solution:

Let us assume that $\neg(x)\,((\exists y)(H(y) \land N(x,y)) \to (\exists y)(A(y) \land N(x,y)))$ as additional premise.

1. $\neg(x)\,((\exists y)(H(y) \land N(x,y)) \to (\exists y)(A(y) \land N(x,y)))$ Additional premise
2. $(\exists x)\neg\,((\exists y)(H(y) \land N(x,y)) \to (\exists y)(A(y) \land N(x,y)))$ 1, *De Morgan's law*
3. $\neg\,((\exists y)(H(y) \land N(a,y)) \to (\exists y)(A(y) \land N(a,y)))$ 2, *ES*
4. $\neg\,(\neg(\exists y)(H(y) \land N(a,y)) \lor (\exists y)(A(y) \land N(a,y)))$ T, 3, *Equivalence*
5. $(\exists y)(H(y) \land N(a,y)) \land \neg(\exists y)(A(y) \land N(a,y))$ 4, *De Morgan's law*
6. $(\exists y)\big(H(y) \land N(a,y)\big)$ T, 5, *conjuction*
7. $H(b) \land N(a,b)$ *ES*, 6
8. $H(b)$ T, 7, *conjuction*
9. $N(a,b)$ T, 7, *conjuction*
10. $\neg(\exists y)(A(y) \land N(a,y))$ T, 5, *conjuction*
11. $(y)(\neg A(y) \lor \neg N(a,y))$ 10, *De Morgan's law*
12. $(\neg A(b) \lor \neg N(a,b))$ 11, *US*
13. $A(b) \to \neg N(a,b)$ T, 12, *Equivalence*
14. $\neg A(b)$ T, 9,13, *Modus tollens*
15. $H(b) \land \neg A(b)$ T, 8,14, *conjuction*
16. $\neg\big(\neg H(b) \lor A(b)\big)$ 15, *De Morgan's law*
17. $\neg\big(H(b) \to A(b)\big)$ T, 16, *Equivalence*
18. $(x)\big(H(x) \to A(x)\big)$ *Rule P*
19. $H(b) \to A(b)$ *US*, 18
20. $\neg\big(H(b) \to A(b)\big) \land \big(H(b) \to A(b)\big)$ T, 17,19 *and conjuction*
21. F *Rule* T, 20 *and negation law*

Prove that $(\exists x)A(x) \to (x)B(x) \Rightarrow (x)(A(x) \to B(x))$

Solution:

Let us assume that $\neg(x)(A(x) \to B(x))$ as additional premise.

1. $\neg(x)\big(A(x) \to B(x)\big)$ Additional premise
2. $(\exists x)\neg\big(A(x) \to B(x)\big)$ 1, *De Morgan's law*
3. $\neg\big(A(a) \to B(a)\big)$ 2, *ES*
4. $(\exists x)A(x) \to (x)B(x)$ *Rule P*
5. $A(a) \to B(a)$ 4, *ES*
6. $\neg\big(A(a) \to B(a)\big) \land A(a) \to B(a)$ T, 3, 5 *and conjuction*
7. F *Rule* T, 6 *and negation law*

Obtain the following implication.

$(x)(P(x) \to Q(x)), (x)(R(x) \to \neg Q(x)) \Rightarrow (x)(R(x) \to \neg P(x))$

Solution:

1. $(x)(P(x) \to Q(x))$ $\quad$ $Rule\ \ P$
2. $P(a) \to Q(a)$ $\quad$ $2, US$
3. $(x)(R(x) \to \neg Q(x))$ $\quad$ $Rule\ \ P$
4. $R(a) \to \neg Q(a)$ $\quad$ $Rule\ US$,3
5. $Q(a) \to \neg R(a)$ $\quad$ Rule T, 4
6. $P(a) \to \neg R(a)$ $\quad$ $T, 2{,}5, hypothetical\ syllogism$
7. $R(a) \to \neg P(a)$ $\quad$ Rule T ,6
8. $(x)\big(R(x) \to \neg P(x)\big)$ $\quad$ $Rule\ UG, 7$