B.E./B.Tech. DEGREE EXAMINATION, APRIL/MAY 2011
Fifth Semester
Computer Science and Engineering
MA2265 – DISCRETE MATHEMATICS
(Regulation 2008)

Part - A

1. Without using truth table show that $\mathbf{P \to (Q \to P) \Rightarrow \sim P \to (P \to Q)}$

Sol: $P \to (Q \to P) \Rightarrow \sim P \to (P \to Q)$

To prove: $\big(P \to (Q \to P)\big) \to \big(\sim P \to (P \to Q)\big)$ is a tautology

$$\Leftrightarrow \sim \big(\sim P \vee (\sim Q \vee P)\big) \vee \big(\sim\sim P \vee (\sim P \vee Q)\big)$$
$$\Leftrightarrow \sim \big(\sim P \vee (\sim Q \vee P)\big) \vee \big(P \vee (\sim P \vee Q)\big)$$
$$\Leftrightarrow \sim \big(\sim P \vee (P \vee \sim Q)\big) \vee \big((P \vee \sim P) \vee Q\big)$$
$$\Leftrightarrow \sim \big((P \vee \sim P) \vee \sim Q\big) \vee (T \vee Q)$$
$$\Leftrightarrow \sim (T \vee \sim Q) \vee T$$
$$\Leftrightarrow T$$

$\therefore \big(P \to (Q \to P)\big) \to \big(\sim P \to (P \to Q)\big)$ is a tautology

$$\therefore P \to (Q \to P) \Rightarrow \sim P \to (P \to Q)$$

2. Show that $\mathbf{\big(P \to (Q \to R)\big) \to \big((P \to Q) \to (P \to R)\big)}$ is a tautology.

Solution:

Let S: $P \to (Q \to R) \to \big((P \to Q) \to (P \to R)\big)$

A: $P \to (Q \to R)$

B: $(P \to Q) \to (P \to R)$

$\mathbf{P}$	Q	R	$\mathbf{P \to Q}$	$\mathbf{P \to R}$	$\mathbf{Q \to R}$	A	$\mathbf{B}$	S
T	T	T	T	T	T	T	T	T
T	F	T	F	T	T	T	T	T
F	T	T	T	T	T	T	T	T
F	F	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	F	F	F	T	T	T	T
F	T	F	T	T	F	T	T	T
F	F	F	T	T	T	T	T	T

Since all the values in last column are true. $\big(P \to (Q \to R)\big) \to \big((P \to Q) \to (P \to R)\big)$ is a tautology.

3. If seven colours are used to paint 50 bicycles, then show that at least 8 bicycles will be the same colour.

Solution:

By Pigeon principle,

If there are n pigeons and k holes, then there is at least one hole contains at least $\left\lfloor \frac{k-1}{n} \right\rfloor + 1$ pigeons.

Here $k = 50, n = 7$

$$\left\lfloor \frac{k-1}{n} \right\rfloor + 1 = \left\lfloor \frac{50-1}{7} \right\rfloor + 1 = \left\lfloor \frac{49}{7} \right\rfloor + 1 = 7 + 1 = 8$$

∴There is at least 8 bicycles will be the same colour.

4. Solve the recurrence relation $\boldsymbol{y(n) - 8y(n-1) + 16y(n-2) = 0}$ for $\boldsymbol{n \geq 2}$, where $\boldsymbol{y(2) = 16}$ and $\boldsymbol{y(3) = 80}$.

Solution:

$$y(n) - 8y(n-1) + 16y(n-2) = 0 \ldots (1)$$

Let $y_n = r^n$ be the solution of (1).

$$(1) \Rightarrow r^n - 8r^{n-1} + 16r^{n-2} = 0$$

$$r^n \left[1 - \frac{8}{r} + \frac{16}{r^2} = 0\right]$$

$$\frac{r^n}{r^2}[r^2 - 8r + 16 = 0]$$

The characteristic equation is $r^2 - 8r + 16 = 0$

$$(r-4)^2 = 0 \Rightarrow r = 4{,}4$$

Hence the solution to this recurrence relation is

$$y_n = \alpha_1 4^n + \alpha_2 n 4^n \ldots (2)$$

$$y_2 = 16 \Rightarrow \alpha_1 + 2\alpha_2 = 1 \ldots (3)$$

$$y_3 = 80 \Rightarrow \alpha_1 + 3\alpha_2 = \frac{5}{4} \ldots (4)$$

Solving (3) and (4), we get

$$\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{4}$$

Substituting $\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{4}$ in (2),

$$y_n = \frac{1}{2}4^n + \frac{1}{4}n4^n$$

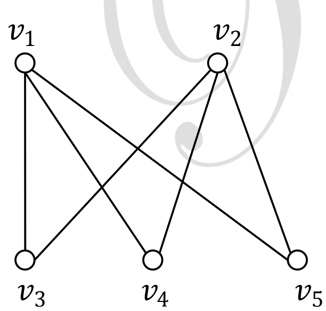
5. Define Pseudo graph.

Answer:

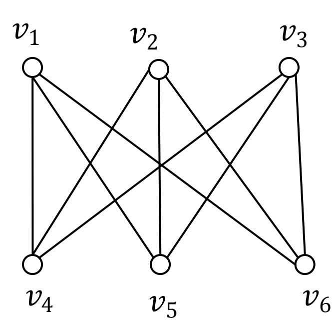
A graph with self loops and parallel edges is called Pseudo graphs.

6. **Draw a complete bipartite graph of $\boldsymbol{K_{2,3}}$ and $\boldsymbol{K_{3,3}}$.**

Solution:



$K_{2,3}$



$K_{3,3}$

7. If a and b are any two elements of a group $(G,*)$, show that G is an Abelian group if and only if $(a*b)^2 = a^2 * b^2$.

Proof:

Let $(G,*)$ be a group and $(a*b)^2 = a^2 * b^2$

$$\Rightarrow (a*b)*(a*b) = (a*a)*(b*b)$$
$$\Rightarrow a*(b*a)*b = a*(a*b)*b \; [Associative\ law]$$
$$\Rightarrow b*a = a*b \; [Left\ and\ Right\ cancellation\ laws]$$
$\Rightarrow (G,*)$ is abelian.

Conversely, let $(G,*)$ be abelian.

$$(a*b)^2 = (a*b)*(a*b)$$
$$= a*(b*a)*b [Associative\ law]$$
$$= a*(a*b)*b \; [\because G\ is\ abelian]$$
$$= (a*a)*(b*b) [Associative\ law]$$
$$\therefore (a*b)^2 = a^2 * b^2$$

8. Let $\langle M,*,e_M \rangle$ be a monoid and $a \in M$. If a is invertible, then show that its inverse is unique.

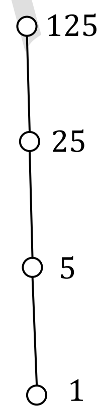
Proof:

Let $a_1 \in M$ and $a_2 \in M$ are the two distinct inverses for $a \in M$.

$$a_1 * a = a * a_1 = e_M \dots (1)$$
$$a_2 * a = a * a_2 = e_M \dots (2)$$
$$a_1 * a = a_2 * a \;\; [from\ (1)\ and\ (2)]$$
$$a_1 = a_2 \; [Right\ cancellation\ law]$$

9. Check whether the posets $\{(1,3,6,9), D\}$and $\{(1,5,25,125), D\}$ are lattices or not. Justify your claim.

Solution:



$\{(1,3,6,9), D\}$ is not a lattice, since $LUB\{6,9\}$ does not exist.

$\{(1,5,25,125), D\}$ is a lattice, since every pair of elements in it has both LUB and GLB.

10. Show that in a Boolean algebra $ab' + a'b = 0$ if and only if $a = b$.

Proof:

Let $a = b$

$$\Rightarrow ab' + a'b = bb' + aa'$$
$$\Rightarrow ab' + a'b = 0 + 0$$
$$\Rightarrow ab' + a'b = 0$$

Conversely, $ab' + a'b = 0$

$$\Rightarrow a + ab' + a'b = a + 0$$
$$\Rightarrow a + a'b = a$$
$$\Rightarrow (a + a')(a + b) = a$$
$$\Rightarrow 1(a + b) = a \Rightarrow (a + b) = a \dots (1)$$
$$ab' + a'b = 0 \ \Rightarrow ab' + a'b + b = 0 + b$$
$$\Rightarrow ab' + b = b$$
$$\Rightarrow (a + b)(b' + b) = b$$
$$\Rightarrow (a + b)1 = b \Rightarrow (a + b) = b$$
$$a + b = b \dots (2)$$

From (1) and (2) we get

$$a = b$$

11.a) i) Use indirect method of proof to prove that
$(\boldsymbol{x})\ (\boldsymbol{P}(\boldsymbol{x}) \vee \boldsymbol{Q}(\boldsymbol{x}))\ \Rightarrow (\boldsymbol{x})\boldsymbol{P}(\boldsymbol{x}) \vee\ (\exists \boldsymbol{x})\boldsymbol{Q}(\boldsymbol{x})$

Solution:

Let us prove this by indirect method

Let us assume that $\neg\big((x)P(x) \vee\ (\exists x)Q(x)\big)$ as additional premise

1. $\neg\big((x)P(x) \vee\ (\exists x)Q(x)\big)$	$Additional\ premise$
2. $\neg(x)\ P(x) \wedge\ \neg\ (\exists x)Q(x)$	$1, De\ Morgan's\ law$
3. $\neg(x)\ P(x)$	$Rule\ T, 2$
4. $(\exists x)\neg\ P(x)$	$3, De\ Morgan's\ law$
5. $\neg\ P(a)$	$Rule\ ES, 4$
6. $\neg\ (\exists x)Q(x)$	$Rule\ T, 2$
7. $(x)\neg\ Q(x)$	$6, De\ Morgan's\ law$
8. $\neg Q(a)$	$Rule\ US, 7$
9. $\neg\ P(a) \wedge \neg Q(a)$	$5{,}8, conjunction$
10. $\neg\ \big(P(a) \vee Q(a)\big)$	$9, De\ Morgan's\ law$
11. $(x)\ (P(x) \vee (Q(x))$	$Rule\ P$
12. $P(a) \vee Q(a)$	$Rule\ US, 11$
13. $\neg\ \big(P(a) \vee Q(a)\big) \wedge \big(P(a) \vee Q(a)\big)$	$11{,}12, conjunction$
14. F	$Rule\ T, 13$

ii) Without using truth table find the PCNF and PDNF of
$\mathbf{P} \rightarrow (\mathbf{Q} \wedge \mathbf{P}) \wedge \big(\sim \mathbf{P} \rightarrow (\sim \mathbf{Q} \wedge \sim \mathbf{R})\big)$

Solution: Let $\mathrm{S} \Leftrightarrow \mathrm{P} \rightarrow (\mathrm{Q} \wedge \mathrm{P}) \wedge \big(\sim \mathrm{P} \rightarrow (\sim \mathrm{Q} \wedge \sim \mathrm{R})\big)$

$\Leftrightarrow \sim \mathrm{P} \vee (\mathrm{Q} \wedge \mathrm{P}) \wedge \big(\sim\sim \mathrm{P} \vee (\sim \mathrm{Q} \wedge \sim \mathrm{R})\big)$

$\Leftrightarrow \sim \mathrm{P} \vee (\mathrm{Q} \wedge \mathrm{P}) \wedge \big(\mathrm{P} \vee (\sim \mathrm{Q} \wedge \sim \mathrm{R})\big)$

$\Leftrightarrow (\sim \mathrm{P} \vee \mathrm{Q}) \wedge (\sim \mathrm{P} \vee \mathrm{P}) \wedge (\mathrm{P} \vee \sim \mathrm{Q}) \wedge (\mathrm{P} \vee \sim \mathrm{R})$

$\Leftrightarrow (\sim \mathrm{P} \vee \mathrm{Q}) \wedge \mathrm{T} \wedge (\mathrm{P} \vee \sim \mathrm{Q}) \wedge (\mathrm{P} \vee \sim \mathrm{R})$

$\Leftrightarrow (\sim \mathrm{P} \vee \mathrm{Q}) \wedge (\mathrm{P} \vee \sim \mathrm{Q}) \wedge (\mathrm{P} \vee \sim \mathrm{R})$

$\Leftrightarrow (\sim \mathrm{P} \vee \mathrm{Q} \vee \mathrm{F}) \wedge (\mathrm{P} \vee \sim \mathrm{Q} \vee \mathrm{F}) \wedge (\mathrm{P} \vee \mathrm{F} \vee \sim \mathrm{R})$

$\Leftrightarrow \big(\sim \mathrm{P} \vee \mathrm{Q} \vee (\mathrm{R} \wedge \sim \mathrm{R})\big) \wedge \big(\mathrm{P} \vee \sim \mathrm{Q} \vee (\mathrm{R} \wedge \sim \mathrm{R})\big) \wedge (\mathrm{P} \vee (\mathrm{Q} \wedge \sim \mathrm{Q}) \vee \sim \mathrm{R})$

$\Leftrightarrow (\sim \mathrm{P} \vee \mathrm{Q} \vee \mathrm{R}) \wedge (\sim \mathrm{P} \vee \mathrm{Q} \vee \sim \mathrm{R}) \wedge (\mathrm{P} \vee \sim \mathrm{Q} \vee \mathrm{R}) \wedge (\mathrm{P} \vee \sim \mathrm{Q} \vee \sim \mathrm{R})$
$\wedge (\mathrm{P} \vee \mathrm{Q} \vee \sim \mathrm{R}) \quad \wedge (\mathrm{P} \vee \sim \mathrm{Q} \vee \sim \mathrm{R})$

$\Leftrightarrow (\sim \mathrm{P} \vee \mathrm{Q} \vee \mathrm{R}) \wedge (\sim \mathrm{P} \vee \mathrm{Q} \vee \sim \mathrm{R}) \wedge (\mathrm{P} \vee \sim \mathrm{Q} \vee \mathrm{R}) \wedge (\mathrm{P} \vee \sim \mathrm{Q} \vee \sim \mathrm{R})$
$\wedge (\mathrm{P} \vee \mathrm{Q} \vee \sim \mathrm{R})$ is a PCNF

$\sim \mathrm{S} \ \Leftrightarrow$ Remaining maxterms in S

$\sim S \Leftrightarrow (P \vee Q \vee R) \wedge (\sim P \vee \sim Q \vee R) \wedge (\sim P \vee \sim Q \vee \sim R)$
$\sim\sim S \Leftrightarrow (\sim P \wedge \sim Q \wedge \sim R) \vee (P \wedge Q \wedge \sim R) \vee (P \wedge Q \wedge R)$ is a PDNF.

b)i) Show that: $(P \to Q) \wedge (R \to S), (Q \wedge M) \wedge (S \to N), \sim (M \wedge N)$ and $P \to R \Rightarrow \sim P$.

Solution:

1. $(P \to Q) \wedge (R \to S)$ — $Rule\ P$
2. $(Q \wedge M) \wedge (S \to N)$ — $Rule\ P$
3. $\sim (M \wedge N)$ — $Rule\ P$
4. $P \to R$ — $Rule\ P$
5. $P \to Q$ — $Rule\ T, 1, P \wedge Q \Rightarrow P$
6. $R \to S$ — $Rule\ T, 1, P \wedge Q \Rightarrow Q$
7. $Q \wedge M$ — $Rule\ T, 2, P \wedge Q \Rightarrow P$
8. $S \to N$ — $Rule\ T, 2, P \wedge Q \Rightarrow Q$
9. $\sim M \vee \sim N$ — $Rule\ T, 3, De\ Morgan's\ law$
10. $P \to S$ — $Rule\ T, 4,6, Chain\ rule$
11. $P \to N$ — $Rule\ T, 10,8, Chain\ rule$
12. M — $Rule\ T, 7, P \wedge Q \Rightarrow Q$
13. $\sim N$ — $Rule\ T, 9,12, Disjunctive\ Syllogism$
14. $\sim P$ — $Rule\ T, 11,13, Modus\ tollens$

ii) Verify that validating of the following inference.
If one person is more successful than another, then he has worked harder to deserve success.
Ram has not worked harder than Siva. Therefore, Ram is not more successful than Siva.
Solution:

Let $S(x,y): x\ is\ more\ successful\ than\ y.$
$H(x,y): x\ works\ harder\ than\ y.$

If one person is more successful than another, then he has worked harder to deserve success.

$$\exists x \exists y (S(x,y) \to H(x,y))$$

Ram has not worked harder than Siva.

$$\sim H(a,b) \quad [where\ a\ is\ Ram\ and\ b\ is\ Siva\ ]$$

Ram is not more successful than Siva.

$$\sim S(a,b)$$

The inference is

$$\exists x \exists y (S(x,y) \to H(x,y)), \sim H(a,b) \Rightarrow \sim S(a,b)$$

1. $\exists x \exists y (S(x,y) \to H(x,y))$ — $Rule\ P$
2. $\sim H(a,b)$ — $Rule\ P$
3. $\exists y (S(a,y) \to H(a,y))$ — $Rule\ T, 1, ES$
4. $S(a,b) \to H(a,b)$ — $Rule\ T, 1, ES$
5. $\sim S(a,b)$ — $Rule\ T, 2,4, Modus\ tollens$

12.a)i) Use Mathematical induction show that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution:

$$Let\ P(n): 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \qquad \dots (1)$$

$P(1): 1^2 = \frac{1(1+1)(2+1)}{6}$

$1 = \frac{6}{6} \Rightarrow 1 = 1$

$\therefore P(1)$ is true.

Let us assume that $P(n)$ is true. Now we have to prove that $P(n+1)$ is true.

To prove:

$$P(n+1): 1^2 + 2^2 + 3^2 + \cdots + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \qquad (from\ (1))$$

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$$

$$= \frac{(n+1)[n(2n+1) + 6(n+1)]}{6}$$

$$= \frac{(n+1)[2n^2 + n + 6n + 6]}{6}$$

$$= \frac{(n+1)[2n^2 + 7n + 6]}{6}$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

$\therefore P(n+1)$ is true.

$\therefore$ By induction method,

$$P(n): 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \text{ is true for all positive integers.}$$

ii) There are 2500 students in a college, of these 1700 have taken a course in C, 1000 have taken a course in Pascal and 550 have taken a course in Networking. Further 750 have taken courses in both C and Pascal. 400 have taken courses in both C and Networking, and 275 have taken courses in both Pascal and Networking. If 200 of these students have taken courses in C, Pascal and Networking.

(1) How many of these 2500 students have taken a course in any of these three courses C, Pascal and Networking?

(2) How many of these 2500 students have not taken a course in any of these three courses C, Pascal and Networking?

Solution:

Let U denote the number of students in a college.

Let A denote the number of students taken a course in C.

Let B denote the number of students taken a course in PASCAL.

Let C denote the number of students taken a course in Networking.

$|U| = 2500, |A| = 1700, |B| = 1000, |C| = 550, |A \cap B| = 750, |A \cap C| = 400,$

$|B \cap C| = 275, |A \cap B \cap C| = 200$

(1) The number of students has taken a course in any of these three courses C, Pascal and Networking

We know that

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$= 1700 + 1000 + 550 - 750 - 400 - 275 + 200$

$|A \cup B \cup C| = 2025.$

(2) The number of students has not taken a course in any of these three courses C, Pascal and Networking is
$|(A \cup B \cup C)'| = |U| - |(A \cup B \cup C)| = 2500 - 2025 = 475$

b) i) Using generating function solve $y_{n+2} - 5y_{n+1} + 6y_n = 0$, $n \geq 0$ with $y_0 = 1$ and $y_1 = 1$.

Solution:

$Let\ G(x) = \sum_{n=0}^{\infty} y_n x^n \ldots (1)$ where $G(x)$ is the generating function for the sequence $\{y_n\}$.

Given $y_{n+2} - 5y_{n+1} + 6y_n = 0$

Multiplying by x_n and summing from 0 to ∞, we have

$$\sum_{n=0}^{\infty} y_{n+2}x^n - 5\sum_{n=0}^{\infty} y_{n+1}x^n + 6\sum_{n=0}^{\infty} y_n x^n = 0$$

$$\frac{1}{x^2}\sum_{n=0}^{\infty} y_{n+2}x^{n+2} - \frac{5}{x}\sum_{n=0}^{\infty} y_{n+1}x^{n+1} + 6\sum_{n=0}^{\infty} y_n x^n = 0$$

$$\frac{1}{x^2}(G(x) - y_1 x - y_0) - \frac{5}{x}(G(x) - y_0) + 6G(x) = 0 \qquad [from\ (1)]$$

$$G(x)\left(\frac{1}{x^2} - \frac{5}{x} + 6\right) - \frac{y_1}{x} - \frac{y_0}{x^2} + \frac{5y_0}{x} = 0$$

$$G(x)\left(\frac{1}{x^2} - \frac{5}{x} + 6\right) - \frac{1}{x} - \frac{1}{x^2} + \frac{5}{x} = 0 \Rightarrow G(x)\left(\frac{6x^2 - 5x + 1}{x^2}\right) = \frac{1}{x^2} - \frac{4}{x}$$

$$G(x)\left(\frac{6x^2 - 5x + 1}{x^2}\right) = \frac{1 - 4x}{x^2}$$

$$G(x) = \frac{1 - 4x}{6x^2 - 5x + 1} = \frac{1 - 4x}{(3x - 1)(2x - 1)}$$

$$\frac{1 - 4x}{(3x - 1)(2x - 1)} = \frac{A}{(3x - 1)} + \frac{B}{(2x - 1)}$$

$1 - 4x = A(2x - 1) + B(3x - 1) \ldots (2)$

Put $x = \frac{1}{2}$ in (2)

$$1 - 4\left(\frac{1}{2}\right) = B\left(\frac{3}{2} - 1\right) \Rightarrow \frac{1}{2}B = -1 \Rightarrow B = -2$$

Put $x = \frac{1}{3}$ in (2)

$$1 - 4\left(\frac{1}{3}\right) = A\left(\frac{2}{3} - 1\right) \Rightarrow -\frac{1}{3}A = -\frac{1}{3} \Rightarrow A = 1$$

$$G(x) = \frac{1}{(3x - 1)} - \frac{2}{(2x - 1)} = -\frac{1}{(1 - 3x)} + \frac{2}{(1 - 2x)}$$

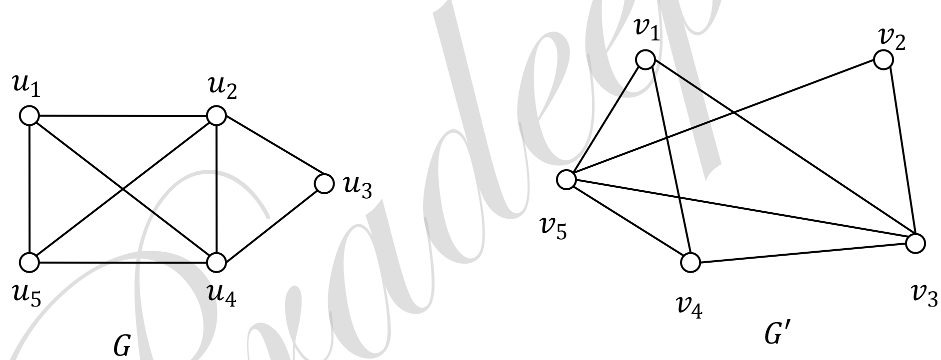
$$\sum_{n=0}^{\infty} y_n x^n = -\sum_{n=0}^{\infty} 3^n x^n + 2\sum_{n=0}^{\infty} 2^n x^n$$

y_n = Coefficient of x^n in $G(x)$

$y_n = -3^n + 2^{n+1}$

ii) A box contains six white balls and five red balls. Find the number of ways four balls can be drawn from the box if
(1) They can be any colour
(2) Two must be white and two red
(3) They must all be the same colour.
Solution:
Total number of balls $= 6 + 5 = 11$
(1) The number of ways four balls can be drawn from the box if they can be any colour is
$$11C_4 = \frac{11 \times 10 \times 9 \times 8}{4!} = 330$$
(2) The number of ways four balls can be drawn from the box if two must be white and two red
$$6C_2 \times 5C_2 = \frac{6 \times 5}{2!} \times \frac{5 \times 4}{2!} = 15 \times 10 = 150$$
(3) The number of ways four balls can be drawn from the box if they must all be the same colour.
$$6C_4 + 5C_4 = 6C_2 + 5C_1 = \frac{6 \times 5}{2!} + 5 = 15 + 5 = 20$$

13. a) i) Examine whether the following pair of graphs are isomorphic. If not isomorphic, give the reasons.



Solution:
In G, the number of vertices is 5, the number of edges is 8.
$$\deg(u_1) = 3 , \deg(u_2) = 4 , \deg(u_3) = 2 , \deg(u_4) = 4 , \deg(u_5) = 3$$
In G', the number of vertices is 5, the number of edges is 8.
$$\deg(v_1) = 3 , \deg(v_2) = 2 , \deg(v_3) = 4 , \deg(v_4) = 3 , \deg(v_5) = 4$$
There are same number of vertices and edges in both the graph G and G'.
Here in both graphs G and G', two vertices are of degree 3, two vertices are of degree 4, and one vertex is of degree 2.
$$u_1 \to v_1, u_2 \to v_5, u_3 \to v_2, u_4 \to v_3, u_5 \to v_4$$
There is one to one correspondences between the graphs G and G'.
∴ The graphs $G$and G' are isomorphic.

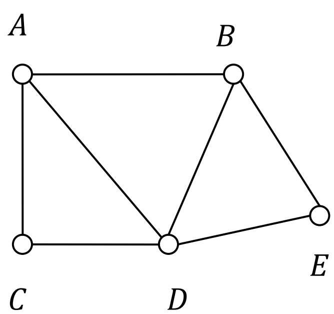
ii) Let G be a simple undirected graph with n vertices. Let u and v be two non adjacent vertices in G such that $\deg(u) + \deg(v) \geq n$ in . Show that G is Hamiltonian if and only if $G + uv$ is Hamiltonian.
Solution:

If G is Hamiltonian, then obviously $G + uv$ is Hamiltonian.
Conversely, suppose that $G + uv$ is Hamiltonian, but G is not.
[**Dirac theorem :** If G is a simple graph with at least three vertices and $\delta(G) \geq \frac{|V(G)|}{2}$ then G is Hamiltonian.]
Then by Dirac theorem, we have

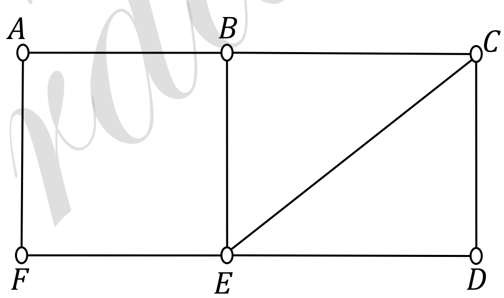
$$\deg(u) < \frac{n}{2} \; and \deg(v) < \frac{n}{2}$$
$$\therefore \deg(u) + \deg(v) < \frac{n}{2} + \frac{n}{2}$$
$$\therefore \deg(u) + \deg(v) < n$$

which is a contradiction to our assumption $G$is not Hamiltonian.
$\therefore$ G is Hamiltonian.
Thus $G + uv$ is Hamiltonian implies G is Hamiltonian.

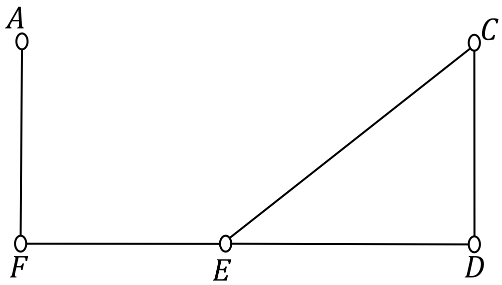
b)i) Draw the graph with 5 vertices, A, B, C, D, E such that $deg(A) = 3$, B is an odd vertex, $deg(C) = 2$ and $D$and E are adjacent.

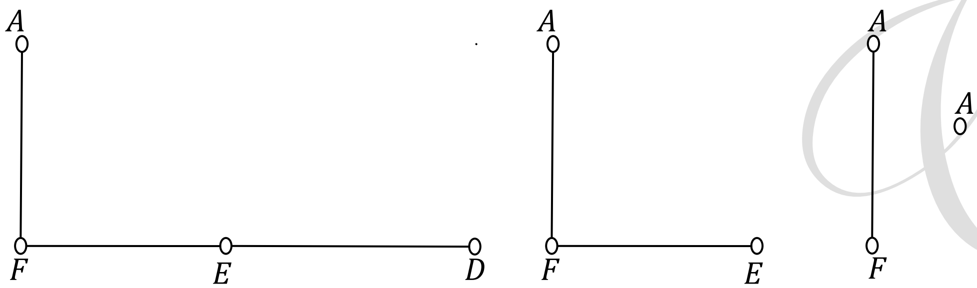


ii) Find the all the connected sub graph obtained form the graph given in the following Figure, by deleting each vertex. List out the simple paths from A to in each sub graph.



Solution:
The connected sub graph obtained form the graph given in the Figure, by deleting each vertex are





The simple paths from $\boldsymbol{A}$ to in each sub graph is

(1) $A \to F \to E \to D \to C,\ A \to F \to E \to C \to D$

(2) $A \to F \to E \to D$

(3) $A \to F \to E$

(4) $A \to F$

14.a) i) If $*$ is a binary operation on the set R of real numbers defined by $\boldsymbol{a * b = a + b + 2ab}$,

(1) Show that $(\boldsymbol{R},*)$ is a semigroup ,

(2) Find the identity element if it exists

(3) Which elements has inverse and what are they?

Solution:

(1) i) Closure: $\forall a, b \in R, a + b + 2ab \in R \Rightarrow a * b \in R$

$\therefore R$ is closed under binary operation $*$.

ii) Associative: $\forall a, b, c \in R$,

$$a * (b * c) = a * (b + c + 2bc)$$
$$= a + (b + c + 2bc) + 2a(b + c + 2bc) = a + b + c + 2ab + 2bc + 2ac + 4abc$$
$$= (a + b + 2ab) + 2(a + b + 2ab)c + c$$
$$= (a + b + 2ab) * c = (a * b) * c$$
$$\therefore a * (b * c) = a * (b + c + 2bc)$$

$\therefore R$ is associative under binary operation $*$.

iii) Identity: Let $e \in R$ be the identity element in R

$$\forall a \in R, a * e = e * a = a$$
$$a + e + 2ae = a \Rightarrow e + 2ae = 0 \Rightarrow e = 0 \in R$$

$\therefore 0 \in R$ is the identity element.

$\therefore (R,*)$ is a semigroup.

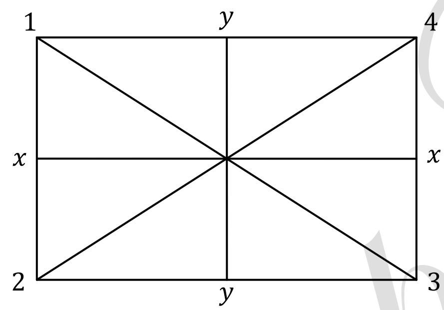
(2) $0 \in R$ is the identity element.

(3) Let $a' \in R$ be the inverse element of $a \in R$

$$\forall a \in R, a * a' = a' * a = e$$
$$a + a' + 2aa' = 0 \Rightarrow a'(1 + 2a) = -a \Rightarrow a' = -\frac{a}{1 + 2a} \in R$$

$\therefore a' = -\frac{a}{1+2a} \in R - \left\{\frac{1}{2}\right\}$ is the inverse element for $\forall a \in R - \left\{\frac{1}{2}\right\}$.

ii) **Define the Dihedral group $(D_4,*)$ and give its composition table. Hence find the identity element and inverse of each element.**
Solution:
The set of transformations due to all rigid motions of a square resulting in identical squares but with different vertex names under the binary operation of right composition $*$ is a group, called dihedral group of order 8 and denoted by $(D_4,*)$.
By rigid motion, we mean the rotation of the square about its centre through angles $90^o, 180^o, 270^o, 360^o$ in the anticlockwise direction and reflection of the square about 4 lines of symmetry is as given in the figure below.



$$r_1 = r(90^o) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix},\ r_2 = r(180^o) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$
$$r_3 = r(270^o) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix},\ r_4 = r(360^o) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$
$$r_5 = r(xx) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},\ r_6 = r(yy) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$
$$r_7 = r(1\,3) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix},\ r_8 = r(2\,4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

The composition table is given below

$*$	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8
r_1	r_2	r_3	r_4	r_1	r_8	r_7	r_5	r_6
r_2	r_3	r_4	r_1	r_2	r_6	r_5	r_8	r_7
r_3	r_4	r_1	r_2	r_3	r_7	r_8	r_6	r_5
r_4	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8
r_5	r_7	r_6	r_8	r_5	r_4	r_2	r_1	r_3
r_6	r_8	r_5	r_7	r_6	r_2	r_4	r_3	r_1
r_7	r_6	r_8	r_5	r_7	r_3	r_1	r_4	r_2
r_8	r_5	r_7	r_6	r_8	r_1	r_3	r_2	r_4

Here $r_4 * r_k = r_k, k = 1,2,3,\ldots,8$.
$\therefore r_4$ is the identity element of $(D_4,*)$.
The inverses of $r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8$ are respectively $r_3, r_2, r_1, r_4, r_5, r_6, r_7$ and r_8.

b) i) Let $(G,*)$ and (H,Δ) be groups and $g: G \rightarrow H$ be a homomorphism. Then prove that kernel of g is a sub-group of G.
Solution:
Let $K = ker(g) = \{g(a) = e' \backslash a \in G, e' \in H\}$
To prove K is a subgroup of G:
We know that $g(e) = e' \Rightarrow e \in K$

$\therefore K$ is a non-empty subset of G.
By the definition of homomorphism $g(a * b) = g(a)\,\Delta\, g(b), \forall a, b \in G$
Let $a, b \in K \Rightarrow g(a) = e^{'}$ and $g(b) = e^{'}$

Now $g(a * b^{-1}) = g(a)\,\Delta\, g(b^{-1}) = g(a)\,\Delta\,\left(g(b)\right)^{-1} = e^{'}\Delta\,(e^{'})^{-1}$
$$= e^{'}\Delta\, e^{'} = e'$$
$$\therefore a * b^{-1} \in K$$
$\therefore K$ is a subgroup of G

ii) State and Prove Lagrange's theorem for finite groups.
Statement:
The order of a subgroup of a finite group is a divisor of the order of the group.
Proof:
Let aH and bH be two left cosets of the subgroup $\{H,*\}$ in the group $\{G,*\}$.
Let the two cosets aH and bH be not disjoint.
Then let c be an element common to aH and bH i.e., $c \in aH \cap\ bH$
$$\because c \in aH, c = a * h_1, for\ some\ h_1 \in H \ldots (1)$$
$$\because c \in bH, c = b * h_2, for\ some\ h_2 \in H \ldots (2)$$
From (1) and (2), we have
$$a * h_1 = b * h_2$$
$$a = b * h_2 * h_1^{-1} \ldots (3)$$
Let x be an element in aH
$x = a *\ h_3, for\ some\ h_3 \in H$
$$= b * h_2 * h_1^{-1} *\ h_3, using\ (3)$$
Since H is a subgroup, $h_2 * h_1^{-1} *\ h_3 \in H$
Hence, (3) means $x \in bH$
Thus, any element in aH is also an element in bH. $\therefore\ aH \subseteq bH$
Similarly, we can prove that $bH \subseteq aH$
Hence $aH = bH$
Thus, if aH and bH are disjoint, they are identical.
The two cosets aH and bH are disjoint or identical. …(4)
Now every element $a \in G$ belongs to one and only one left coset of H in G,
For,
$a = ae \in aH, since\ e \in H \Rightarrow a \in aH$
$a \notin bH$, since aH and bH are disjoint i.e., a belongs to one and only left coset of H in G i.e., $aH \ldots (5)$
From (4) and (5), we see that the set of left cosets of H in G form the partition of G. Now let the order of H be m.
Let $H = \{h_1, h_2, \ldots, h_m\}, where\ h_i{}'s$ are distinct
Then $aH = \{ah_1, ah_2, \ldots, ah_m\}$
The elements of aH are also distinct, for, $ah_i = ah_j \Rightarrow h_i = h_j$, which is not true.
Thus H and aH have the same number of elements, namely m.
In fact every coset of H in G has exactly m elements.
Now let the order of the group $\{G,*\}$ be n, i.e., there are n elements in G
Let the number of distinct left cosets of H in G be p.
$\therefore$ The total number of elements of all the left cosets $= pm$ $=$ the total number of elements of G. i.e., $n = pm$

i.e., m, the order of H is adivisor of n, the order of G.

15. a) i) Show that every distributive lattice is a modular. Whether the converse is true? Justify your answer

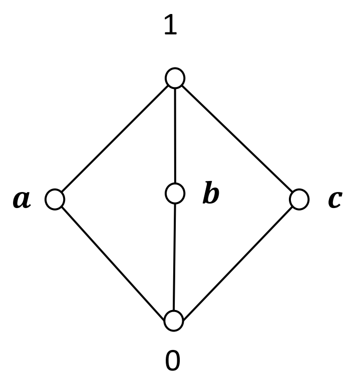
Solution:

Let $a, b, c \in L$ and assume that $a \le c, then$

$$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$$
$$= (a \oplus b) * c$$

∴Every distributive lattice is modular.

For example let us consider the following lattice



Here in this Lattice,

$$\forall a, b, c \in L, a \le b \Rightarrow a \oplus (b * c) = (a \oplus b) * c$$

∴The above lattice is modular.

$$a * (b \oplus c) = a * 1 = a \ldots (1)$$
$$(a * b) \oplus (a * c) = 0 \oplus 0 = 0 \ldots (2)$$

From (1) and (2) we get $a * (b \oplus c) \ne (a * b) \oplus (a * c)$

∴The above lattice is not distributive.

∴ Every distributive lattice is a modular but its converse is not true.

ii) **Prove that the direct product of any two distributive lattices is a distributive lattice.**

Solution:

Let $(L, *, \oplus)$ and $(S, \wedge, \vee)$ $be\ two\ distributive\ lattices\ and\ let$ $(L \times S, ., +)$ $be\ the\ direct\ product\ of\ two\ lattices.$

For any $(a_1, b_1), (a_2, b_2)$ and $(a_3, b_3) \in L \times S$

$$(a_1, b_1).\big((a_2, b_2) + (a_3, b_3)\big) = (a_1, b_1).(a_2 \oplus a_3, b_2 \vee b_3)$$
$$= \big(a_1 * (a_2 \oplus a_3), b_1 \wedge (b_2 \vee b_3)\big)$$
$$= \big((a_1 * a_2) \oplus (a_1 * a_3), (b_1 \wedge b_2) \vee (b_1 \wedge b_3)\big)$$
$$= (a_1, b_1).(a_2, b_2) + (a_1, b_1).(a_3, b_3)$$

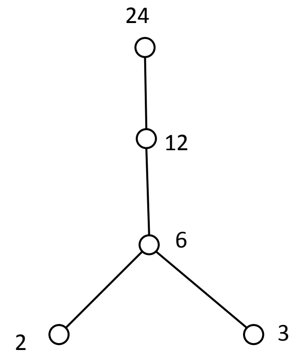
∴The direct product of any two distributive lattices is a distributive lattice.

b)i) Draw the Hasse diagram for

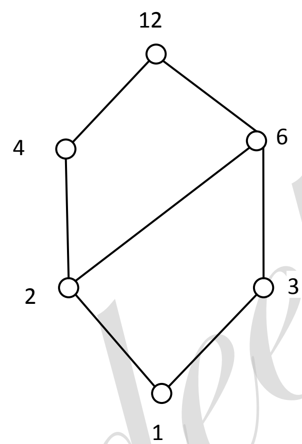
$(\mathbf{1})$ $\boldsymbol{P_1} = \{\mathbf{2, 3, 6, 12, 24}\}$ $(\mathbf{2})$ $\boldsymbol{P_2} = \{\mathbf{1, 2, 3, 4, 6, 12}\}$ ***and ≤ is a relation be such that*** $\boldsymbol{x \le y}$ $\boldsymbol{iff\ x \mid y}$.

Solution:

The Hasse diagram for P_1 is



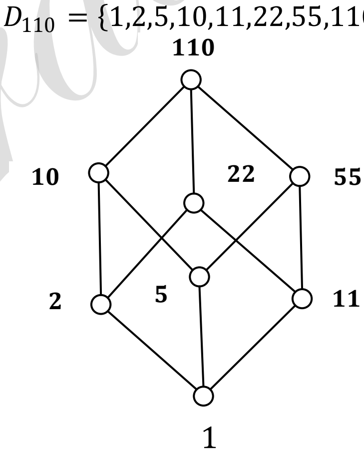
The Hasse diagram for P_2 is



ii) Prove that D_{110} , the set of all positive divisors of a positive integer 110, is a Boolean algebra 110 and find all its sub algebras.

Solution:

$$D_{110} = \{1,2,5,10,11,22,55,110\}$$



Since set all divisors D satisfies reflexive, anti-symmetric and transitive properties, D is a partial order relation.

$\therefore (D_{110}, D)$ is a Poset.

From the Hasse diagram, we observe that every element in the Poset (D_{110}, D) has a least upper bound and greatest lower bound. $\therefore (D_{110}, D)$ is a Lattice.

Here 1 is the least element and 110 is the greatest element.

From the Hasse diagram, we observe that $\forall\ a, b, c \in D_{110}, a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
$\therefore$ D_{110} is a distributive Lattice.
The complement of 1 is 110. $[\because 1 \wedge 110 = 1\ \&\ 1 \vee 110 = 110\ ]$
The complement of 2 is 55. $[\because 2 \wedge 55 = 1\ \&\ 2 \vee 55 = 110\ ]$
The complement of 5 is 22. $[\because 5 \wedge 22 = 1\ \&\ 22 \vee 5 = 110\ ]$
The complement of 10 is 11. $[\because 10 \wedge 11 = 1\ \&\ 10 \vee 11 = 110\ ]$
The complement of 11 is 10.
The complement of 22 is 5.
The complement of 55 is 2.
The complement of 110 is 1.
$\because$ Every element in D_{110} has atleast one complement, D_{110} is a complemented Lattice.
The sub Boolean algebras are
$i)$ {1,110}
$ii)$ {1,2,5,10,11,22,55,110}
$iii)$ {1,2,5,110}
$iv)$ {1,2,11,110}
$v)$ {1,5,11,110}
$vi)$ {1,10,22,110}
$vii)$ {1,10,55,110}
$viii)$ {1,22,11,110}