B.E / B.Tech Degree Examinations, April/May 2011

MA2261 Probability and Random process. S. KANAKALAKS

Answer ALL Questions

Part A – (10 × 2 = 20 Marks)

1. The CDF of a continuous random variable is given by $F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x/5}, & 0 \le x < \infty \end{cases}$ Find the PDF and mean of X.

The pdf of X is $f(x) = \frac{d}{dx} F(x)$

$$\frac{d}{dx}(1 - e^{-x/5}) = 0 - e^{-x/5} \times (-1/5)$$

$$\therefore f(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{5} e^{-x/5}, & 0 \le x < \infty. \end{cases}$$

$$\text{Mean} = E[X] = \begin{cases} \sum_i x_i p_i & \text{if } X \text{ is discrete RV} \\ \int_{-\infty}^{\infty} x f(x)\,dx & \text{if } X \text{ is continuous RV.} \end{cases}$$

$$\therefore \text{Mean} = \int_0^\infty \frac{1}{5} \underset{u}{x} \underset{v}{e^{-x/5}} dx = \frac{1}{5}\left[x\left(\frac{e^{-x/5}}{-1/5}\right) - (1)\left(\frac{e^{-x/5}}{(-1/5)^2}\right)\right]_0^\infty = \frac{1}{5}\left[\{0-0\} - \{-\cancel{25}\right.$$

$$= \frac{1}{5}[25 + 5(0)] = \underline{5 + 0} = 5$$

2. Establish the memoryless property of the exponential distribution.

$P(X > S+t / X > t) = P(X > S)$, for any $s, t > 0$.

The pdf of X is $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$

$$\therefore P(X > k) = \int_k^\infty \lambda e^{-\lambda x} dx = \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_k^\infty = \lambda\left[0 + \frac{e^{-\lambda k}}{\lambda}\right) = e^{-\lambda k}.$$

$$P(X > (s+t) / X > t) = \frac{P(X > s+t \cap X > t)}{P(X > t)} = \frac{P(X > s+t)}{P(X > t)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = \frac{e^{-\lambda s} \cdot e^{-\lambda t}}{e^{-\lambda t}}$$

$$= e^{-\lambda s} = P(X > s)$$

$\therefore P(X > s+t / X > t) = P(X > s)$ ∴ Exponential distribution possesses memoryless property.

3. Let X and Y be continuous random variables with joint probability density function $f_{XY}(x,y) = \frac{x(x-y)}{8}$, $0<x<2$, $-x<y<x$ and $f_{XY}(x,y)=0$ elsewhere. Find $f_{Y/X}(y/x)$.

$f_{Y/X}(y/x) = \frac{f(x,y)}{f(x)}$, $f(x)$ = Marginal density function of $X = \int_{-x}^{x} f(x,y)dy$

$f(x) = \int_{-x}^{x} \frac{x^2-xy}{8} dy = \frac{1}{8}\left[x^2y - x\frac{y^2}{2}\right]_{-x}^{x} = \frac{1}{8}\left[x^2(x+x) - \frac{x}{2}\underbrace{(x^2-x^2)}_{0}\right] = \frac{2x^3}{8} =$

$\therefore f_{Y/X}(y/x) = \frac{\frac{x(x-y)}{8}}{x^3/4} = \frac{x(x-y)}{8_2} \times \frac{4}{x^{3}_{2}} = \frac{x-y}{2x^2}$ //. ; $0<x<2$, $-x<y<x$.

4. Find the acute angle between the two lines of regression, assuming the two lines of regression.

The angle between the regression lines is given by $\tan\theta = \frac{1-\rho^2}{\rho} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2+\sigma_y^2}$

5. Prove that a first order stationary process has a constant mean.

The first order density of a stationary process $\{X(t)\}$ is independent of time t

i.e $f(x_1, t_1+c) = f(x_1, t_1)$

$\Rightarrow E[X(t)] = \mu$ = a constant.

i.e $E[X(t)]$ is a constant in a first order stationary R.P.

$\therefore$ A first order stationary process has a constant mean.

6. State the postulates of a poisson process.

If $X(t)$ represents the number of occurrences of certain event in $(0,$ then the discrete random process $\{X(t)\}$ is called the poisson process, provid the following postulates are satisfied.

(i) $P[0$ occurrence in $(t, t+\Delta t)] = 1-\lambda\Delta t$

(ii) $P[1$ occurrence in $(t, t+\Delta t)] = \lambda\Delta t$.

(iii) $P[2$ or more occurrence in $(t, t+\Delta t)] = 0$.

(iv) $X(t)$ is independent of the number of occurrences of the even any Interval prior and after the interval $(0,t)$.

v) The probability that the event occurs a specified number of times (t₀, t₀+t) depends only on t, but not on t₀.

7. The autocorrelation function of a stationary random process is $R(\tau) = 16 + \dfrac{9}{1+16\tau^2}$. Find the mean and variance of the process.

By properties of Autocorrelation function,

$$\bar{X}^2 = \lim_{\tau \to \infty} R_{XX}(\tau) = \lim_{\tau\to\infty} 16 + \frac{9}{1+16\tau^2} = 16 \Rightarrow \bar{X} = 4. \Rightarrow \mu_1' = 4.$$

$$E(X^2(t)) = \lim_{\tau\to 0} R_{XX}(\tau) = \lim_{\tau\to 0}\left(16 + \frac{9}{1+16\tau^2}\right) = 16 + \frac{9}{1} = 25 \Rightarrow \mu_2' = 25$$

$\therefore$ Mean $= E[X(t)] = \bar{x} = \mu = \mu_1' = 4.$

Variance $= \mu_2' - \mu_1'^2 = 25 - 4^2 = 25 - 16 = 9.$

8. Prove that for a WSS process $\{X(t)\}$, $R_{XX}(t, t+\tau)$ is an even function of τ.

By definition, $R(\tau) = E[X(t)X(t+\tau)]$

Put $t+\tau = t_1 \Rightarrow t = t_1 - \tau.$

$$\therefore R(\tau) = E[X(t_1 - \tau)X(t_1)]$$
$$= E[X(t_1) \cdot X(t_1 + (-\tau))]$$

$$R(\tau) = R(-\tau)$$

$\therefore$ The autocorrelation function is an even function of τ.

9. Find the system Transfer function, if a Linear Time Invariant system has an impulse function $H(t) = \begin{cases} \frac{1}{2c}, & |t| \le c \\ 0, & |t| \ge c. \end{cases}$

The system transfer function $H(\omega) = F[H(t)] = F[\frac{1}{2c}]$

$$H(\omega) = \int_{-\infty}^{\infty} H(t)dt = \int_{-c}^{c} \frac{1}{2c}\, dt = \frac{1}{2c}\int_{-c}^{c} dt = \frac{1}{2c}[t]_{-c}^{c} = \frac{1}{2c}(c-(-c))$$

$$\therefore H(\omega) = \frac{1}{2c}(2c) = 1.$$

10. Define white Noise.

A noise process whose power spectral density is independent of the operating frequency is called white noise.

PART B

11. a) The probability density function of a random variable x is given by

$$f_X(x) = \begin{cases} x, & 0 < x < 1 \\ k(2-x), & 1 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

(1) Find the value of k

(2) Find $p(0.2 < x < 1.2)$

(3) What is $p[0.5 < x < 1.5 / x > 1]$

(4) Find the distribution function of f(x).

(1) By total probability, $\int_{-\infty}^{\infty} f(x)\,dx = 1$ i.e $\int_{-\infty}^{0} 0\,dx + \int_{0}^{1} x\,dx + \int_{1}^{2} k(2-x)\,dx + \int_{2}^{\infty}$

i.e $\left(x^2/2\right)_0^1 + k\left[2x - x^2/2\right]_1^2 = 1$

$\frac{1}{2} + k\left[(4 - \frac{1}{2}) - (2 - \frac{1}{2})\right] = 1 \Rightarrow \frac{1}{2} + k\left[4 - 2 - \frac{1}{2} + \frac{1}{2}\right] =$

$\frac{1}{2} + 2k = 1 \Rightarrow 2k = 1 - \frac{1}{2} \Rightarrow 2k = \frac{1}{2} \Rightarrow \boxed{k = \frac{1}{4}}$

(2) $p(0.2 < x < 1.2)$

$= \int_{0.2}^{1.2} f(x)\,dx = \int_{0.2}^{1} x\,dx + \int_{1}^{1.2} \frac{1}{4}(2-x)\,dx$

$= \left[\frac{x^2}{2}\right]_{0.2}^{1} + \frac{1}{4}\left[2x - \frac{x^2}{2}\right]_1^{1.2}$

$= \left[\frac{1}{2} - \frac{(0.2)^2}{2}\right] + \frac{1}{4}\left[\left\{2(1.2) - \frac{(1.2)^2}{2}\right\} - \left\{2 - \frac{1}{2}\right\}\right]$

$= \frac{1}{2} - 0.02 - \frac{1}{4}\left[2.4 - 0.72 - 2 + \frac{1}{2}\right]$

$= \frac{1}{2} - 0.02 - 0.6 + 0.18 + \frac{1}{2} - \frac{1}{8} = \frac{1 - 0.62 + 0.18 - 1/8}{0.56 - 0.125}$

$P(0.2 < x < 1.2) = 0.435$

(3) $p(0.5 < x < 1.5 / x > 1) = \frac{p[0.5 < x < 1.5 \cap x > 1]}{p[x > 1]}$

$= \frac{p(1 < x < 1.5)}{p(x > 1)}$

$\therefore p(1<x<1.5) = \int_1^{1.5} \frac{1}{4}(2-x)dx = \frac{1}{4}\left[2x - \frac{x^2}{2}\right]_1^{1.5}$

$= \frac{1}{4}\left[2(1.5-1) - \frac{1}{2}(1.5^2-1^2)\right] = \frac{1}{4}\left[2(0.5) - \frac{1}{2}(1.25)\right]$

$p(1<x<1.5) = 0.09375$

$p(x>1) = \int_1^{\infty} f(x)dx = \int_1^{2} \frac{1}{4}(2-x)dx = \frac{1}{4}\left[2x - \frac{x^2}{2}\right]_1^2$

$= \frac{1}{4}\left[2(2-1) - \frac{1}{2}(4-1)\right] = \frac{1}{4}\left[2 - \frac{3}{2}\right] = 0.125$

$\therefore p(0.5<x<1.5 / x\geq 1) = \frac{0.09375}{0.125} = 0.75$

(4) The distribution function of $X = F(x) = \int_{-\infty}^{x} f(x)dx$.

When $0<x<1$, $F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{x} f(x)dx = 0 + \int_0^x x\,dx = \frac{x^2}{2}$

when $1\leq x<2$, $F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^1 f(x)dx + \int_1^x f(x)dx$.

$F(x) = 0 + \int_0^1 x\,dx + \int_1^x \frac{1}{4}(2-x)dx = \left(\frac{x^2}{2}\right)_0^1 + \frac{1}{4}\left[2x - \frac{x^2}{2}\right]_1^x = \frac{1}{2} + \frac{1}{4}\left[(2x \ldots\right.$
$\left. - (2- \ldots\right.$

$F(x) = \frac{1}{2} + \frac{1}{4}\left(2x - \frac{x^2}{2}\right) - \frac{1}{4}\left(2 - \frac{1}{2}\right) = \cancel{\frac{1}{2}} + \frac{1}{4}\left(2x - \frac{x^2}{2}\right) - \cancel{\frac{1}{2}} + \frac{1}{8}$

$F(x) = \frac{1}{4}\left(2x - \frac{x^2}{2}\right) + \frac{1}{8}$.

when $x>2$, $F(x) = \int_{-\infty}^{0} f(x)dx + \int_0^1 f(x)dx + \int_1^2 \frac{1}{4}(2-x)dx + \int_2^{\infty} f(x)dx$.

$F(x) = 0 + \int_0^1 x\,dx + \int_1^2 (2x - x)dx + 0 = 1$.

$$\therefore F(x) = \begin{cases} \frac{x^2}{2}, & 0<x\leq 1 \\ \frac{1}{4}\left(2x - \frac{x^2}{2}\right) + \frac{1}{8}, & 1\leq x<2, \\ 0, & x>2. \end{cases}$$

11) b) i) Derive the m.g.f of Poisson distribution and hence or otherwise deduce its mean and variance.

The probability mass function of poisson distribution is

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x=0,1,2,\ldots$$

The moment generating function of the Poisson distribution is

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^{x}}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(e^{t}\lambda)^{x}}{x!} = e^{-\lambda}e^{\lambda e^{t}} = e^{\lambda(e^{t}-1)}$$

To find mean and variance using MGF

$$\text{Mean} = E(X) = \mu_1' = \left[\frac{d}{dt}(M_X(t))\right]_{t=0} = \left[\frac{d}{dt}\left(e^{\lambda(e^{t}-1)}\right)\right]_{t=0}$$

$$\mu_1' = \left[e^{-\lambda}e^{\lambda e^{t}}\cdot\lambda e^{t}\right]_{t=0} = e^{-\lambda}e^{\lambda}\cdot\lambda = \lambda. \text{ i.e } \boxed{E(X(t)) = \lambda}$$

$$\mu_2' = E(X^2) = \left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0} = \left[\frac{d^2}{dt^2}\left(e^{-\lambda}e^{\lambda e^{t}}\right)\right]_{t=0}$$

$$= \left[e^{-\lambda}e^{\lambda e^{t}}(\lambda e^{t})^2 + e^{-\lambda}e^{\lambda e^{t}}\lambda e^{t}\right]_{t=0} = e^{-\lambda}e^{\lambda}\lambda^2 + e^{-\lambda}e^{\lambda}\lambda = \lambda^2+\lambda$$

$$\text{Variance} = E(X^2) - [E(X)]^2 = \lambda+\lambda^2-\lambda^2 = \lambda. \Rightarrow \boxed{Var(X) = \lambda}$$

ii) The marks obtained by a number of students in a certain subject are assumed to be normally distributed with mean 65 and standard deviation 5. If 3 students are selected at random from this set, what is the probability that exactly 2 of them will have marks over 70?

Given $\mu = 65, \sigma = 5$ $\quad z = \frac{X-\mu}{\sigma} = \frac{X-65}{\sigma=5}$

when $X=70$, $z = \frac{70-65}{5} \Rightarrow z = 1.$

$\therefore P(X>70) = P(Z>1) = 0.5 - P(0<Z<1) = 0.5 - 0.3413 = \underbrace{0.158}_{P}$

$\therefore q = 1-P = 0.8413$

$\therefore$ P (No. of students i.e 2 of them scoring more than 70)

$$= P(Y=2) = nC_y\, p^{y} q^{n-y}, \quad n=3.$$

$$P(Y=2) = 3C_2\, p^2 q^1 = 3\times(0.1587)^2(0.8413) = 0.06358.$$

12) a) i) If X and Y are independent poisson random variables with respective parameters λ_1 and λ_2. Calculate the conditional distribution of X, given that $X+Y=n$.

Given X and Y are independently and poisson distributed random variables.

$$P(X=r/X+Y=n) = \frac{P(X=r \cap X+Y=n)}{P(X+Y=n)} = \frac{P(X=r)\cdot P(\cancel{X+}Y=n\cancel{-X})}{P(X+Y=n)} \quad (\because X=r)$$

$$= \frac{P(X=r \cap Y=n-r)}{P(X+Y=n)} = \frac{P(X=r)\cdot P(Y=n-r)}{P(X+Y=n)}$$

$$= \frac{\left(\frac{e^{-\lambda_1}\lambda_1^r}{r!}\right)\left(\frac{e^{-\lambda_2}\lambda_2^{n-r}}{(n-r)!}\right)}{e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!}} = \frac{\cancel{e^{-(\lambda_1+\lambda_2)}}\lambda_1^r\lambda_2^{n-r}}{\cancel{e^{-(\lambda_1+\lambda_2)}}(\lambda_1+\lambda_2)^n} \times \frac{n!}{r!(n-r)!}$$

$$= \frac{n!}{r!(n-r)!}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^r\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-r}$$ (Since $X+Y$ is a poisson variate with parameter $\lambda_1+\lambda_2$)

$= nC_r\, p^r q^{n-r}$ which is a pmf of a binomial distribution.

where $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$, $q = \frac{\lambda_2}{\lambda_1+\lambda_2}$ and $nC_r = \frac{n!}{r!(n-r)!}$.

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ii) The regression equation of X on Y is $3Y-5X+108=0$. If the mean value of Y is 44 and the variance of X is $\frac{9}{16}$th of the variance of Y. Find the mean value of X and the correlation coefficient.

Given the line of regression of x on y is $5x = 3y+108$

i.e $x = \frac{3}{5}y + 108/5$

i.e $\boxed{x = 0.6y + 21.6} \Rightarrow b_{xy} = 0.6$.

Also given $\bar{y} = 44$. and $Var(x) = \frac{9}{16} Var(y)$.

$$\therefore b_{yx} = \rho\frac{\sigma_y}{\sigma_x} = \rho\frac{\sigma_y}{\sqrt{\frac{9}{16}\sigma_y^2}} = \rho\frac{4\cancel{\sigma_y}}{3\cancel{\sigma_y}}$$

$$\Rightarrow b_{yx} = 4\rho. \quad \Rightarrow \rho = \frac{b_{yx}}{4}.$$

$$\rho = \sqrt{b_{xy} \times b_{yx}}$$

$$\frac{b_{yx}}{4} = \sqrt{0.6 * b_{yx}}$$

$$\frac{(b_{yx})^{\cancel{2}}}{16} = 0.6\,\cancel{b_{yx}}$$

$b_{yx} = 16 \times 0.6 \Rightarrow \boxed{b_{yx} = 9.6.}$

$\therefore \rho = \sqrt{9.6 \times 0.6} =$

12. b) i) If x and y are independent random variables with density function $f_X(x) = \begin{cases} 1, & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$ and $f_Y(y) = \begin{cases} y/6, & 2 \le y \le 4 \\ 0, & \text{otherwise} \end{cases}$, find the density function of $Z = XY$.

Given X and Y are independent random variables

$$\therefore f(x,y) = f_X(x) \cdot f_Y(y) = \begin{cases} 1 \cdot y/6, & 1 \le x \le 2,\ 2 \le y \le 4. \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \frac{y}{6}, & 2 \le y \le 4,\ 1 \le x \le 2. \\ 0, & \text{otherwise.} \end{cases}$$

Given $z = xy$ let $w = y$

$x = \frac{z}{y} = \frac{z}{w}$ and $y = w$

$$J = \begin{vmatrix} \partial x/\partial z & \partial x/\partial w \\ \partial y/\partial z & \partial y/\partial w \end{vmatrix} = \begin{vmatrix} \frac{1}{w} & -z/w^2 \\ 0 & 1 \end{vmatrix} = \frac{1}{w}$$

The joint PDF of z and w is given by

$$f_{ZW}(z,w) = f_{XY}(z,w)\,|J| = \frac{y}{6} \cdot \frac{1}{w} = \begin{cases} \frac{w}{6} \cdot \frac{1}{w}, & 1 \le \frac{z}{w} \le 2 \ \& \ 2 \le w \le 4 \\ 0, & \text{other} \end{cases}$$

i.e $f_{ZW}(z,w) = \begin{cases} \frac{1}{6}, & 2 \le w \le 4 \;\&\; 1 \le z \le 2w \\ 0, & \text{otherwise} \end{cases}$

Range:

$2 \le y \le 4 \Rightarrow 2 \le w \le 4$

$1 \le x \le 2 \Rightarrow 1 \le \frac{z}{w} \le 2 \Rightarrow 1 \le z \le 2w$

b) ii) The life time of a particular variety of electric bulb may be considered as a random variable with mean 1200 hours and standard deviation 250 hours. Using central limit theorem, find the probability that the average life time of 60 bulbs exceeds 1250 hours.

Let $\bar{x}$ denote the mean life-time of 60 bulbs.

Given $n=60$ and $\mu = 1200$, $\sigma = 250$.

$\bar{x}$ follows normal distribution with mean μ and variance $\frac{\sigma^2}{n}$.

To find the probability that average life time of 60 bulbs exceeds 1250 hours:

$$P(\bar{x} \ge 1250) = P\left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \ge \frac{1250-\mu}{\sigma/\sqrt{n}}\right) = P\left(z \ge \frac{1250-1200}{250/\sqrt{60}}\right)$$

$$= P(z \ge 1.55) = P(0 \le z \le \infty) - P(0 \le z \le 1.55)$$

$$= 0.5 - 0.4394$$

$$\underline{P(\bar{x} \ge 1250)} = 0.0606.$$

13) a) i) A random process $X(t)$ defined by $X(t) = A\cos t + B\sin t$, $-\infty < t < \infty$, where A and B are independent random variables each of which takes a value -2 with probability $1/3$ and a value 1 with probability $2/3$. Show that $X(t)$ is wide-sense stationary.

$\{X(t)\}$ is WSS if $E[X(t)] =$ constant and $R_{XX}(t, t+\tau)$ depends only on τ.

Since ~~A&B are~~ discrete R.V's which assumes values -2 and 1 with probability respectively.

$\therefore$ Mean $E(A) = \sum a_i p(a_i) = (-2)\frac{1}{3} + 1(\frac{2}{3}) = 0 = E(B)$.

$E(A^2) = \sum a_i^2 p(a_i) = (-2)^2(\frac{1}{3}) + (1)^2(\frac{2}{3}) = \frac{4}{3} + \frac{2}{3} = 2 = E(B^2)$

Also given A and B are independent RV's.

$\therefore E(AB) = E(A) \cdot E(B) = 0$.

Since $X(t) = A\cos t + B\sin t$

(i) $E[X(t)] = E[A\cos t] + E[B\sin t]$

$= \cos t\, E(A) + \sin t\, E(B) = \cos t(0) + \sin t(0)$

$= 0$, a constant.

(ii) By definition, $R_{XX}(t, t+\tau) = E[X(t) \cdot X(t+\tau)] = E[(A\cos t + B\sin t) \times (A\cos(t+\tau) + B\sin(t+\tau))]$

$= E[A^2 \cos t \cos(t+\tau)] + E[AB\{\cos t \sin(t+\tau) + \sin t \cdot \cos(t+\tau)\}]$

$+ E(B^2 \sin t \sin(t+\tau))]$

$= E(A^2).\cos t \cos(t+\tau) + E(AB).\sin(2t+\tau) + E(B^2)\sin t \sin(t+\tau)$

$= 2\cos t \cos(t+\tau) + 0 + 2\sin t \sin(t+\tau)$ $(\because E(AB) = E(A)E(B) = 0)$

$= 2[\cos t \cos(t+\tau) + \sin t \sin(t+\tau)]$

$= 2\cos(t+\tau-t)$

$R(\tau) = 2\cos\tau$, which depends on τ alone.

$\therefore \{X(t)\}$ is WSS process.

13)a)(i) A random process has sample functions of the form $X(t) = A\cos(\omega t + \theta)$, where ω is constant, A is a random variable with mean zero and variance 1 and θ is a random variable that is uniformly distributed between 0 and 2π. Assume that random variables A and θ are independent. Is $X(t)$ is mean-ergodic process?

Given $x(t) = A\cos(\omega t + \theta)$

$E(A) = 0$, $Var(A) = 1$. $\Rightarrow E(A^2) - (E(A))^2 = 1$

$$\boxed{E(A^2) = 1.}$$

and $\theta \sim U(0, 2\pi)$. $f(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 < \theta < 2\pi \\ 0, & \text{otherwise.} \end{cases}$

(i) $E(X(t)) = \int_{-\infty}^{\infty} \cancel{A\cos(\omega t+\theta)}$. $E[A\cdot\cos(\omega t+\theta)] = E(A)\cdot E[\cos(\omega t+\theta)]$

$$= \left(\int_{-\infty}^{\infty} \cos(\omega t+\theta) . f(\theta)\, d\theta\right) E(A)$$

$\boxed{E(x(t)) = 0}$, since $E(A) = 0$ —— (1)

(ii) $\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} x(t)\,dt = \frac{1}{2T}\int_{-T}^{T} A\cos(\omega t+\theta)\,.dt$

$$= \frac{A}{2T}\left[\frac{\sin(\omega t+\theta)}{\omega}\right]_{-T}^{T}$$

$$= \frac{A}{2T\omega}\left[\sin(\omega T+\theta) - \sin(-\omega T+\theta)\right]$$

$$= \frac{A}{2T\omega}\left[\sin(\theta+\omega T) - \sin(\theta-\omega T)\right]$$

$\sin(A+B) - \sin(A-B) = 2\cos A \sin B$

$$= \frac{A}{\cancel{2}T\omega}\cdot \cancel{2}\cos\theta \sin\omega T.$$

$$\lim_{T\to\infty} \overline{X}_T = \lim_{T\to\infty} \frac{A}{T\omega}\cos\theta \sin\omega T$$

$\boxed{\lim_{T\to\infty} \overline{X}_T = 0.}$ (since $\frac{\text{Anything}}{\infty} = 0$) —— (2)

From (1) and (2), $\{x(t)\}$ is mean Ergodic.

13) b) (i) If $\{X(t)\}$ is a Gaussian process with $\mu(t) = 10$ and $C(t_1, t_2) = 16 e^{-|t_1 - t_2|}$ find the probability that

(1) $X(10) \le 8$

(2) $|X(10) - X(6)| \le 4$.

$\{X(t)\}$ is a Normal process with parameter with mean $= 10$

WKT $C(t_1, t_2) = R_{XX}(t_1, t_2) - E[X(t_1)]E[X(t_2)]$

$\therefore C(t, t) = R_{XX}(t, t) - \{E[X(t)]\}^2 = E[X^2(t)] - (E(X(t)))^2$

$= \text{Variance}(X(t)) = 16 e^{|t-t|} = 16 \Rightarrow \sigma = 4.$

Case (i) $P[X(10) \le 8] = P\left[\frac{X(10) - \text{mean}}{\sigma} \le \frac{8-10}{4}\right] = P(Z \le -0.5)$

$= 0.5 - P(0 \le Z \le 0.5) = 0.5 - 0.1915$

$= 0.3087.$

Case (ii) Let $X = X(10) - X(6)$ $\therefore E[X(t)] = E[X(10) - X(6)] = 10 - 10 = 0$

$C(t_1, t_2) = Var(X(t_1)) + Var[X(t_2)] - 2 Cov(X(t_1), X(t_2))$

$C(t, t) = 16 + 16 - 2 Cov(t, t) = 16 + 16 - 2(16 e^{-|10-6|}) = 31.4$

$\boxed{\sigma = 5.6}$ $\therefore P[\{|X(10) - X(6)|\} \le 4] = P[|X(t)| \le 4] = P(-4 \le X(t) \le 4)$

$= P(-0.714 \le Z \le 0.714) = 2P(0 < Z < 0.714) = 0.522.$

b) (ii) Prove that the interval between two successive occurrences of a poisson process with parameter λ has an exponential distribution with mean $\frac{1}{\lambda}$.

Let $X(t)$ denote the number of occurrences in a time interval t.

Let T be the interval between two successive occurrences E_i and E_{i+1}

T is a continuous random variable.

Let the event E_i occurs at the time instant t_i.

$P(T > t) = P[E_{i+1}$ did not occur in $(t_i, t_i + t)]$

$= P[$no event occurs in $(t_i, t_i + t)]$

$= P[X(t) = 0] = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}$

$\Rightarrow P(T > t) = e^{-\lambda t}$

$\therefore$ Cumulative distribution function $F(t) = P(T \le t) = 1 - P(T > t)$

$= 1 - e^{-\lambda t}.$

Probability density function $f(t) = \frac{d}{dt} F(t).$

i.e $f(t) = F'(t) = 0 - (-\lambda) e^{-\lambda t} = \lambda e^{-\lambda t}$. which is the PDF of exponential function.

For exponential distribution, mean $= \frac{1}{\lambda}$.

Hence proved.

14) a) i) The power spectral density function of a zero mean WSS process $X(t)$ is given by $S(\omega) = \begin{cases} 1, & |\omega| < \omega_0 \\ 0, & \text{otherwise} \end{cases}$. Find $R(\tau)$ and show that $X(t)$ and $X\left(t+\frac{\pi}{\omega_0}\right)$ are uncorrelated.

Given $E[X(t)] = 0$. By definition $R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega) e^{j\omega\tau} d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega\tau} d\omega$

$$R(\tau) = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} (\cos\omega\tau + j\sin\omega\tau) d\omega = \frac{2}{2\pi}\int_0^{\omega_0} \cos\omega\tau\, d\omega + 0.$$

$$= \frac{1}{\pi}\left[\frac{\sin\omega\tau}{\tau}\right]_0^{\omega_0} = \frac{1}{\pi}\left[\frac{\sin\omega_0\tau}{\tau} - 0\right]$$

$$\therefore R(\tau) = \frac{\sin\omega_0\tau}{\pi\tau}$$

$$C\left[X(t)\, X(t+\pi/\omega_0)\right] = E\left[X(t)\, X\left(t+\frac{\pi}{\omega_0}\right)\right] - E[X(t)]\, E\left[X(t+\pi/\omega_0)\right]$$

$$= R_{XX}(\pi/\omega_0) - 0 = R_{XX}(\pi/\omega_0) \quad \text{since } E[X(t)] = 0.$$

But $R_{XX}(\tau) = \frac{\sin\omega_0\tau}{\pi\tau}$

$$\therefore R_{XX}(\pi/\omega_0) = \frac{\sin\omega_0\left(\frac{\pi}{\omega_0}\right)}{\pi\left(\pi/\omega_0\right)} = \frac{\sin\pi}{\pi^2/\omega_0} = 0.$$

$$\therefore C\left[X(t)\, X\left(t+\frac{\pi}{\omega_0}\right)\right] = 0.$$

$\therefore$ $X(t)$ and $X(t+\pi/\omega_0)$ are uncorrelated.

ii) The ACF of a WSS process is given by $R(\tau) = \alpha^2 e^{-2\lambda|\tau|}$ determine the PSD of the process.

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} \alpha^2 e^{-2\lambda|\tau|} e^{-j\omega\tau} d\tau$$

$$= \alpha^2 F\left[e^{-2\lambda|\tau|}\right]$$

$$= \alpha^2 \frac{2(2\lambda)}{4\lambda^2 + \omega^2} = \frac{4\alpha^2\lambda}{4\lambda^2 + \omega^2}.$$

14)b) i) State and prove Weiner Khintchine theorem.

If $X_T(\omega)$ is the Fourier Transform of the truncated random process defined as $X_T(t) = \begin{cases} X(t), & -T \le t \le T \\ 0, & \text{otherwise} \end{cases}$ where $\{X(t)\}$ is a real WS process with PSD function $S_{XX}(\omega)$ then $S_{XX}(\omega) = \lim_{T\to\infty}\left[\frac{1}{2T} E\{|X_T(\omega)|^2\}\right]$

Given that $X_T(\omega) = F[X_T(t)] = \int_{-\infty}^{\infty} X_T(t) e^{-j\omega t} dt = \int_{-T}^{T} X(t) e^{-j\omega t} dt$

$$|X_T(\omega)|^2 = X_T(\omega)\overline{X_T(\omega)} = X_T(\omega) X_T(-\omega) = \int_{-T}^{T} X(t_1) e^{-j\omega t_1} dt_1 \int_{-T}^{T} X(t_2) e^{+j\omega t_2} dt_2$$

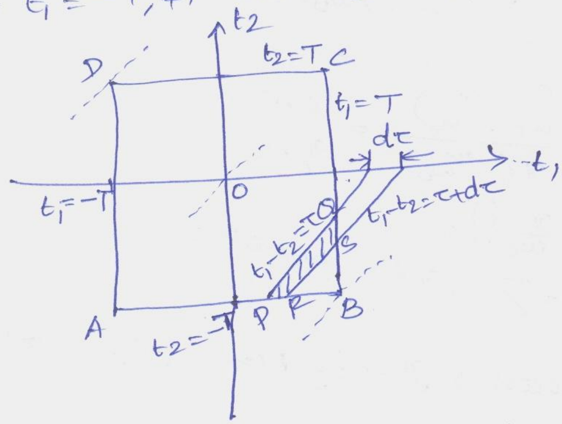
$$= \int_{-T}^{T}\int_{-T}^{T} X(t_1) X(t_2) e^{-j\omega(t_1-t_2)} dt_1 dt_2$$

$$\therefore E\{|X_T(\omega)|^2\} = \int_{-T}^{T}\int_{-T}^{T} E[X(t_1)X(t_2)] e^{-j\omega(t_1-t_2)} dt_1 dt_2$$

$$= \int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1-t_2) e^{-j\omega(t_1-t_2)} dt_1 dt_2 \quad \text{since } R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$$

$$= \int_{-T}^{T}\int_{-T}^{T} \phi(t_1-t_2) dt_1 dt_2, \text{ where } \phi(t_1-t_2) = R_{XX}(t_1-t_2) e^{-j\omega(t_1-t_2)}$$

We evaluate the double integral over the area of the square ABCD bounded by $t_1 = -T, +T$ and $t_2 = -T, T$ as shown in the figure.



Let us divide the area of the square into a number of strips like PQRS, PQ is given by $t_1 - t_2 = \tau$ and RS given by $t_1 - t_2 = \tau + d\tau$.

When PQRS is at the initial position D, $t_1 - t_2 = -2T$. When PQRS is at the final position B, $t_1 - t_2 = 2T$. Hence when τ varies from $-2T$ to $2T$, the area ABCD is covered.

Now $dt_1 dt_2$ = elemental area of the $t_1 t_2$ plane = area of PQRS.

when $\tau > 0$, Area of PQRS = Area of triangle PBQ − Area of Δ RBS

$$= \tfrac{1}{2}(2T-\tau)^2 - \tfrac{1}{2}(2T-\tau-d\tau)^2$$

$$= (2T-\tau)d\tau.$$

$$BQ = PB = T-(\tau-T) = \begin{cases} 2T-\tau, & \tau \geq 0 \\ 2T+\tau, & \tau < 0. \end{cases}$$

For all τ we have Area of PQRS $= dt_1 dt_2 = (2T-|\tau|)d\tau$.

$$\therefore E[|X_T(\omega)|^2] = \int_{-2T}^{2T} \phi(\tau)(2T-|\tau|)d\tau$$

$$\therefore \frac{1}{2T} E[|X_T(\omega)|^2] = \int_{-2T}^{2T} \phi(\tau)\left(1-\frac{|\tau|}{2T}\right)d\tau$$

$$\lim_{T\to\infty} \frac{1}{2T} E[|X_T(\omega)|^2] = \lim_{T\to\infty}\int_{-2T}^{2T}\phi(\tau)d\tau - \lim_{T\to\infty}\frac{1}{2T}\int_{-2T}^{2T}|\tau|\phi(\tau)d\tau$$

$$= \int_{-\infty}^{\infty}\phi(\tau)d\tau, \quad \text{assuming } \int_{-\infty}^{\infty}\phi(\tau)|\tau|d\tau \text{ is bounded}$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}d\tau$$

$$\therefore \lim_{T\to\infty} E[|X_T(\omega)|^2] = S_{XX}(\omega).$$ By the definition of PSD

4)b) ii) The cross-power spectrum of real random process $\{X(t)\}$ and $\{Y(t)\}$ is given by $S_{XY}(\omega) = \begin{cases} a+bj\omega, & |\omega|<1 \\ 0, & \text{elsewhere} \end{cases}$ Find the cross correlation function.

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{j\omega\tau}d\omega = \int_{-1}^{1}(a+jb\omega)e^{j\omega\tau}d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1}(a+jb\omega)\cos\omega\tau\, d\omega + \frac{1}{2\pi}j\int_{-1}^{1}(a+jb\omega)\sin\omega\tau\, d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1}a\cos\omega\tau\, d\tau + \frac{j}{2\pi}b\int_{-1}^{1}\omega\cos\omega\tau\, d\omega + \frac{j}{2\pi}a\int_{-1}^{1}\sin\omega\tau\, d\omega$$

$$+ \frac{j^2 b}{2\pi}\int_{-1}^{1}\omega\sin\omega\tau\, d\omega$$

$$= \frac{2}{2\pi}\int_0^1 a\cos\omega\tau\, d\omega + 0 + 0 - \frac{2b}{2\pi}\int_0^1 \omega\sin\omega\tau\, d\omega$$

$$= \frac{1}{\pi}\left[a\frac{\sin\omega\tau}{\tau} - b\left\{\omega\left(-\frac{\cos\omega\tau}{\tau}\right) - 1\left(-\frac{\sin\omega\tau}{\tau^2}\right)\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{a\sin\tau}{\tau} + \frac{b\cos\tau}{\tau} - \frac{b\sin\tau}{\tau^2}\right]$$

$$R_{xy}(\tau) = \frac{1}{\pi\tau^2}\left[(a\tau - b)\sin\tau + b\tau\cos\tau\right].$$

15a) i) Consider a system with transfer function $\frac{1}{1+j\omega}$. An input signal with autocorrelation function $m\delta(\tau) + m^2$ is fed as input to the system. Find the mean and mean-square value of the output.

Given system transfer function $H(\omega) = \frac{1}{1+j\omega}$

Input autocorrelation $R_{xx}(\tau) = m\delta(\tau) + m^2$.

$$\therefore S_{xx}(\omega) = F[m\delta(\tau) + m^2] = m\,F[\delta(\tau)] + F[m^2] = m + 2\pi m^2\delta(\omega)$$

WKT $S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega) = H(\omega)\cdot H^*(\omega)\cdot S_{xx}(\omega)$

$$= \frac{1}{1+j\omega}\cdot\frac{1}{1-j\omega}\left[m + 2m^2\pi\delta(\omega)\right]$$

$$\underline{\underline{S_{yy}(\omega) = \frac{1}{1+\omega^2}\left[m + 2\pi m^2\delta(\omega)\right]}}$$

Mean of the output $E[y(t)] = \bar{Y} = H(0)\cdot\bar{X}$.

Now $\bar{X}^2 = \lim_{\tau\to\infty} R_{xx}(\tau) = \lim_{\tau\to\infty}[m\delta(\tau) + m^2]$

$\bar{X}^2 = m^2 \Rightarrow \boxed{\bar{X} = m}$ and $H(0) = \frac{1}{0+1} = 1$ and.

$$\bar{Y} = H(0)\cdot\bar{X}$$

$$= m\cdot 1$$

$$\boxed{\bar{Y} = m}$$

Mean square value of the output $= R_{yy}(0)$

$$R_{yy}(\tau) = F^{-1}[S_{yy}(\omega)] = F^{-1}\left[\frac{1}{1+\omega^2}\left(m + 2\pi m^2\delta(\omega)\right)\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+\omega^2}\left(m + 2\pi m^2 \delta(\omega)\right) e^{j\omega\tau} d\omega.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{k e^{j\omega\tau}}{1+\omega^2} d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\pi m^2 \delta(\omega) e^{j\omega\tau}}{1+\omega^2} d\omega$$

$$= \frac{k}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+\omega^2} e^{j\omega\tau} d\omega + k^2 \int_{-\infty}^{\infty} \frac{\delta(\omega)}{1+\omega^2} e^{j\omega\tau} d\omega$$

$$= \frac{k}{2\pi}\, \pi e^{-|\tau|} + k^2$$

Hence $R_{YY}(\tau) = k/2\, e^{-|\tau|} + k^2$.

Mean square value $\overline{Y}^2 = R_{XX}(0) = k/2 + k^2$ //.

a) ii) An LTI System has an impulse response $h(t) = e^{-\beta t} u(t)$. Find the output autocorrelation function $R_{YY}(\tau)$ corresponding to an input $X(t)$.

We know that $H(\omega) = F[h(t)] = \int_0^{\infty} e^{-\beta t} e^{-j\omega t} dt = \int_0^{\infty} e^{-(\beta+j\omega)t} dt$

$$= -\left[\frac{e^{-(\beta+j\omega)t}}{\beta+j\omega}\right]_0^{\infty} = \frac{1}{\beta+j\omega}$$

$$|H(\omega)|^2 = \frac{1}{\beta^2+\omega^2}.$$

The output power density spectrum $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

$$S_{YY}(\omega) = \frac{1}{\beta^2+\omega^2} S_{XX}(\omega). = \frac{1}{2\beta}\left(\frac{2\beta}{\beta^2+\omega^2}\right) S_{XX}(\omega).$$

Taking inverse Fourier transform on bothsides of the above expression,

$$R_{YY}(\tau) = \frac{1}{2\beta} F^{-1}\left[\frac{2\beta}{\beta^2+\omega^2}\right] * R_{XX}(\tau) = \frac{1}{2\beta} e^{-\beta|t|} * R_{XX}(\tau) = \frac{1}{2\beta}\int_0^{\infty} e^{-\beta|u|} R_{XX}(\tau-u)du$$

15) b(i) A linear system is described by the impulse response $h(t) = \frac{1}{RC} e^{-t/RC} u(t)$. Assume an input process whose ACF is $A\delta(\tau)$. Find the mean and ACF of the output process.

Let $\beta = \frac{1}{RC}$.

$\therefore h(t) = \left(\frac{1}{RC} e^{-t/RC}\right) u(t) = \beta e^{-\beta t} u(t)$

we know that $S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$.

$\therefore |H(\omega)|^2 = H(\omega)\overline{H(\omega)}$

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \beta e^{-\beta t} u(t) e^{-j\omega t} dt.$$

$$= \int_{-\infty}^{0} 0\,dt + \int_{0}^{\infty} \beta e^{-\beta t} (1) e^{-j\omega t} dt$$, using the definition of $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$.

$$= \beta \int_0^{\infty} e^{-(\beta + j\omega)t} dt = \beta \left[\frac{e^{-(\beta+j\omega)t}}{-(\beta+j\omega)}\right]_0^{\infty}$$

$$= \beta \left[e^{-\infty} + \frac{e^0}{\beta + j\omega}\right] = \frac{\beta}{\beta + j\omega}.$$

$\therefore H(\omega) = \frac{\beta}{\beta + j\omega}$. and $\overset{*}{H}(\omega) = \overline{H(\omega)} = \frac{\beta}{\beta - i\omega}$; $|H(\omega)|^2 = \frac{\beta^2}{\beta^2 + \omega^2}$

$\therefore S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$

$S_{yy}(\omega) = \frac{\beta^2}{\beta^2 + \omega^2} S_{xx}(\omega)$. ; $S_{xx}(\omega) = F[R_{xx}(\tau)] = F[A\delta(\tau)] = A$.

$\therefore S_{yy}(\omega) = \frac{\beta^2}{\beta^2 + \omega^2} A$

$$\therefore R_{yy}(\tau) = F^{-1}\left[\frac{A\beta^2}{\beta^2 + \omega^2}\right] = A\beta^2 F^{-1}\left(\frac{1}{\omega^2 + \beta^2}\right) = A\beta^2 \frac{1}{2\beta} e^{-\beta|\tau|}$$

$$\therefore R_{yy}(\tau) = \frac{A\beta}{2} e^{-\beta|\tau|}$$

15) b) (ii) If $\{N(t)\}$ is a band-limited white noise such that

$$S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}, & |\omega| < \omega_B \\ 0, & \text{elsewhere.} \end{cases}$$ find the autocorrelation function.

By definition, the PSD is given by

$$S_{NN}(\omega) = \begin{cases} N_0/2, & |\omega| < \omega_B \\ 0, & \text{elsewhere.} \end{cases}$$

The ACF is given by the inverse Fourier transform

$$\therefore R_{NN}(\tau) = \frac{1}{2\pi}\int_{-\omega_B}^{\omega_B} \frac{N_0}{2} e^{j\omega\tau} d\omega = \frac{N_0}{4\pi}\int_{-\omega_B}^{\omega_B} e^{j\omega\tau} d\omega$$

$$= \frac{N_0}{4\pi}\left[\frac{e^{j\omega\tau}}{j\tau}\right]_{-\omega_B}^{\omega_B} = \frac{N_0}{4\pi}\left[\frac{e^{j\omega_B\tau} - e^{-j\omega_B\tau}}{j\tau}\right]$$

$$= \frac{N_0}{2\pi\tau}\left[\frac{e^{j\omega_B\tau} - e^{-j\omega_B\tau}}{2j}\right]$$

$$R_{NN}(\tau) = \frac{N_0}{2\pi}\left(\frac{\sin\omega_B\tau}{\tau}\right).$$