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CHENNAI – APRIL / MAY 2010 MA2261 – PROBABILITY AND RANDOM PROCESSES

Answer All Questions

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Part A – (10 x 2 = 20 marks)

1. If the pdf of a random variable $X$ is $f(x) = \frac{x}{2}$ in $0 \le x \le 2$, find $P(X > 1.5 / X > 1)$

Answer:

Given $f(x) = \frac{x}{2}$ in $0 \le x \le 2$

$$P(X>1.5/X>1) = \frac{P(X>1.5 \text{ and } X>1)}{P(X>1)}$$

$$= \frac{P(X>1.5)}{P(X>1)}$$

$$p(x>1.5/x>1) = \frac{\frac{1.75}{4}}{3/4} = \frac{1.75}{\cancel{4}} \times \frac{\cancel{4}}{3} = \frac{1.75}{3} = \frac{7}{12}$$

where $p(x>1.5) = \int_{1.5}^{2} f(x)dx = \int_{1.5}^{2} x/2\, dx = \frac{1}{2}\left(\frac{x^2}{2}\right)_{1.5}^{2} = \frac{1.75}{4}$

and $p(x>1) = \int_{1}^{2} f(x)dx = \int_{1}^{2} x/2\, dx = 1/2\left(\frac{x^2}{2}\right)_{1}^{2} = 3/4.$

2. If the MGF of a uniform distribution for a random variable $X$ is $\frac{1}{t}(e^{5t} - e^{4t})$ find $E[X](t)$

Answer: Given M.G.F of a uniform distribution, $M_X(t) = \frac{e^{5t} - e^{4t}}{t}$ —— ①

We know that $M_X(t) = \frac{e^{bt} - e^{at}}{(b-a)t}$ —— ②

Comparing ① and ②, $b = 5$ and $a = 4$. $\therefore$ Mean $E(X) = \frac{a+b}{2} = \frac{9}{2}$

3. Find the value of $k$, if $f(x,y) = k(1-x)(1-y)$ in $0 < x < 4$, $1 < y < 5$ and $f(x,y) = 0$, otherwise, is to be the joint density function.

Answer: Since $f(x,y)$ is a p.d.f, we have $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$

i.e $\int_{1}^{5}\int_{0}^{4} k(1-x)(1-y)dxdy = k\int_{1}^{5} (1-y)\left[\frac{(1-x)^2}{-2}\right]_{0}^{4} dy = 1 \Rightarrow \frac{k}{-2}\int_{1}^{5} 8(1-y)dy = 1$

i.e $\frac{8k}{-2}\int_{1}^{5}(1-y)dy = 1 \Rightarrow -4k\left(\frac{(1-y)^2}{-2}\right)_{1}^{5} = 1 \Rightarrow 2k(16-0) = 1 \Rightarrow \boxed{k = \frac{1}{32}}$

4) A random variable X has mean 10 and variance 16. Find the lower bound for $P(5, X<15)$.

Out of Syllabus as it comes under Tchebycheff's inequality.

5. Define a wide sense stationary process.

A random process $\{X(t)\}$ is called a weakly stationary process or covariance stationary process or wide sense stationary process if its mean is a constant and the autocorrelation depends only on the time difference.

i.e., $E[X(t)] = \text{constant}$ and $R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$ = depends only $\tau$.

6. Define a Markov chain and give an example.

Answer: Let $\{X(t)\}$ be a Markov process which possess Markov prope[rty] and which takes only discrete values whether t is discrete or disc[rete]. Then $\{X(t)\}$ is called as Markov chain.

Mathematically, we define the Markov chain as follows.

If $P[X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \ldots, X_0 = a_0] = P[X_n = a_n / X_{n-1} = a_{n-1}]$ for all n then the process $\{X_n\}$, $n = 0, 1, 2, \ldots$ is called as Markov Chai[n].

Here $a_0, a_1, a_2, \ldots, a_n$ are called the states of the Markov Chain.

7. Find the mean of the stationary process $\{X(t)\}$, whose autocorrelation function is given by $R(\tau) = 16 + \frac{9}{1+16\tau^2}$

Answer: Given $R_{XX}(\tau) = 16 + \frac{9}{1+16\tau^2}$

Let us assume that the process be stationary. Then we know that

$$\overline{X}^2 = \lim_{|\tau|\to\infty} R_{XX}(\tau) \quad \text{—— (1)}$$

Substituting $R_{XX}(\tau)$ in (1), we have, $\overline{X}^2 = \lim_{|\tau|\to\infty}\left[16 + \frac{9}{1+16\tau^2}\right] = \lim_{|\tau|\to\infty} 16 + \lim_{|\tau|\to\infty} \frac{9}{1+16\tau^2}$

$\overline{X}^2 = 16 \Rightarrow \overline{X} = 4$ $\therefore \boxed{\text{Mean} = E[X(t)] = 4.}$

8. Find the power spectral density function of the stationary process whose autocorrelation function is given by $e^{-|\tau|}$

Answer: Given Autocorrelation function, $R_{XX}(\tau) = e^{-|\tau|}$

The spectral density function is given by $S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$

2 $S_{XX}(\omega) = \int_{-\infty}^{\infty} e^{-|\tau|} e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} e^{-|\tau|} (\cos\omega\tau - j\sin\omega\tau) d\tau = \int_{-\infty}^{\infty} e^{-|\tau|} \cos\omega\tau \, d\tau - j\int_{-\infty}^{\infty} e^{-|\tau|} \sin\omega$

Since $|\tau|$ is an even fn, $e^{-|\tau|}\sin\omega\tau$ is an odd function and $e^{-|\tau|}\cos\omega\tau$ is an even fn.

$\Rightarrow S_{XX}(\omega) = \int_{-\infty}^{\infty} \underbrace{e^{-|\tau|}}_{\text{even}} \times \underbrace{\cos\omega\tau}_{\text{even}} d\tau - j(0) = 2\int_0^{\infty} e^{-\tau} \cos\omega\tau \, d\tau$ $\left(\because |\tau| = \begin{cases} +\tau, (0, \\ -\tau, (-\infty, \end{cases}\right.$

$S_{XX}(\omega) = 2\left(\frac{1}{1^2+\omega^2}\right) = \frac{2}{1+\omega^2}$.

9. Define time-invariant system.

Answer: Let $Y(t) = f(X(t))$. If $Y(t+h) = f[X(t+h)]$, then $f$ is called a time-invariant system or $X(t)$ and $Y(t)$ are said to form a time-invariant system.

.. State autocorrelation function of the white noise.

Answer

A sample function $x(t)$ of a wide-sense stationary noise random process $X(t)$ called White Noise if the power spectrum of $N(t)$ is a constant at all frequencie

Thus, we define, $S_{XX}(\omega) = \frac{N_0}{2}$, for a white noise where $N_0$ is a real positive constant. By inverse Fourier transform,

$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{N_0}{2} e^{j\omega\tau} d\tau = \frac{N_0}{2}\delta(\tau)$ $\left(\because \int_{-\infty}^{\infty} e^{j\omega} d \ldots = 2\pi\ldots\right.$

Part-B (5×16 = 80 Marks)

11 a) i) The probability mass function of random variable X is defined as $P(X=0) = 3c^2$, $P(X=1) = 4c - 10c^2$, $P(X=2) = 5c - 1$, where $c > 0$, and $P(X=r) = 0$ if $r \neq 0, 1, 2$. Find (1). The value of c (2) $P(0 < X < 2 / X > 0)$ (3) The distribution function of X (4) The largest value of X for which $F(x) < 1/2$.

Solution: We know that the total probability is equal to 1. i.e., $\sum_{x=0}^{r} P(x) = 1$.

(1) $P(X=0) + P(X=1) + P(X=2) = 1 \Rightarrow 3c^2 + 4c - 10c^2 + 5c - 1 = 1$

$\Rightarrow 7c^2 - 9c + 2 = 0 \Rightarrow (7c-2)(c-1) = 0 \Rightarrow c = 2/7$ or $c = 1$

$\boxed{c = 2/7}$ ( c cannot be 1 as all probability will be >1).

(2) $P(0 < X < 2 / X > 0) = \frac{P(0 < X < 2 \cap X > 0)}{P(X > 0)} = \frac{4c - 10c^2}{9c - 10c^2 - 1} = \frac{4(2/7) - 10(4/\ldots}{9(2/7) - 10(4/4\ldots}$

$= 0.4324$.

(3) $F(x) = P(X \le x)$

| $X$ | $F(x) = P(X \le x)$ |
|---|---|
| 0 | $F(0) = P(X=0) = 3c^2 = 3(4/49) = 0.2449$ |
| 1 | $F(1) = P(X=0)+P(X=1) = -7c^2+4c = 0.5714$ |
| 2 | $F(2) = P(X=0)+P(X=1)+P(X=2) = 1.$ |

(4) From the new table $F(x) < 1/2$ for $x=0$.

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11a) (ii) If the probability that an applicant for a driver's license will pass the road test on any given trial is 0.8. What is the probability that he will finally pass the test (1) On the fourth trial and (2) In less than 4 trials?

Solution: $p = 0.8 \quad q = 1-p = 0.2 \;;\; P(X=x) = q^{x-1}p, \; x = 1, 2, \ldots$

$P(\text{Passing in the 4th trial}) = q^3 p = (0.2)^3(0.8) = 0.0064.$

$P(X<4) = P(X=1)+P(X=2)+P(X=3) = q^0 p + qp + q^2 p = p(1+q+q^2)$
$= 0.992$

(OR)

11.b) (i) Find the MGF of the two parameter exponential distribution whose density function is given by $f(x) = \lambda e^{-\lambda(x-a)}, x \ge a$ and hence find the mean and variance.

Out of syllabus

11.b) (ii) The marks obtained by a number of students in a certain subject are assumed to be normally distributed with mean 65 and standard deviation 5. If 3 students are selected at random from this group, what is the probability that atleast one of them would have scored above 75?

Solution:

Let $X$ be the RV which denotes the marks obtained by students.

Given $\mu = 65$, $\sigma = 5$. The standard normal variate $z = \frac{X-\mu}{\sigma} = \frac{X-65}{5}$ — (1)

To find $P(X>75)$

when $X=75$, $z = \frac{75-65}{5} = 2$ (using (1))

$\therefore P(X>75) = P(z>2) = 0.5 - P(0<z<2) = 0.5 - 0.4772 = 0.0228$

$\therefore P(\text{a student scores} > 75) = 0.0228$ i.e $p = 0.0228 \Rightarrow q = 0.9772$

and $n=3$. Let $Y$ be the number of students scoring more than 75.

$\therefore P(Y=y) = nC_y \, p^y q^{n-y}$ (Binomial).

3 $P(Y=y) = 3C_y\,(0.0228)^y (0.9772)^{3-y}$

$\therefore P(Y \geq 1) = 1 - P(Y<1) = 1 - P(Y=0) = 1 - 3C_0\,(0.0228)^0 (0.9772)^3$

$= 1 - (0.9772)^3 = 0.0667.$

2.a) i) For the bivariate probability distribution of $(X, Y)$ given below:

| X \ Y | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1/32 | 2/32 | 2/32 | 3/32 |
| 1 | 1/16 | 1/16 | 1/8 | 1/8 | 1/8 | 1/8 |
| 2 | 1/32 | 1/32 | 1/64 | 1/64 | 0 | 2/64 |

Find the marginal distributions, conditional distribution of X given Y=1 and conditional distribution of Y given X=0.

Solution:

| X \ Y | 1 | 2 | 3 | 4 | 5 | 6 | P(X=x) |
|---|---|---|---|---|---|---|---|
| 0 | 0 P(0,1) | 0 P(0,2) | 1/32 P(0,3) | 2/32 P(0,4) | 2/32 P(0,5) | 3/32 P(0,6) | 8/32 P(X=0) |
| 1 | 1/16 P(1,1) | 1/16 P(1,2) | 1/8 P(1,3) | 1/8 P(1,4) | 1/8 P(1,5) | 1/8 P(1,6) | 10/16 P(X=1) |
| 2 | 1/32 P(2,1) | 1/32 P(2,2) | 1/64 P(2,3) | 1/64 P(2,4) | 0 P(2,5) | 2/64 P(2,6) | 8/64 P(X=2) |
| P(Y=y) | 3/32 P(Y=1) | 3/32 P(Y=2) | 11/64 P(Y=3) | 13/64 P(Y=4) | 6/32 P(Y=5) | 16/64 P(Y=6) | 1 |

Row Total

Column Total

Conditional distribution of X given Y=1.

$$P(X=0/Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{0}{3/32} = 0.$$

$$P(X=1/Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{1/16}{3/32} = 2/3.$$

$$P(X=2/Y=1) = \frac{P(X=2, Y=1)}{P(Y=1)} = \frac{1/32}{3/32} = 1/3.$$

Conditional distribution of Y given X=0,

$$P(Y=1/X=0) = \frac{P(X=0, Y=1)}{P(X=0)} = 0.$$

$$P(Y=2/X=0) = \frac{P(X=0, Y=2)}{P(X=0)} = 0.$$

$$P(Y=3/X=0) = \frac{P(X=0, Y=3)}{P(X=0)} = \frac{1/32}{8/32} = 1/8.$$

$$P(Y=4/X=0) = \frac{P(X=0, Y=4)}{P(X=0)} = \frac{2/32}{8/32} = 1/4.$$

$$P(Y=5/X=0) = \frac{P(X=0, Y=5)}{P(X=0)} = \frac{2/32}{8/32} = 1/4.$$

$$P(Y=6/X=0) = \frac{P(X=0, Y=6)}{P(X=0)} = \frac{3/32}{8/32} = 3/8.$$

12.a) ii) Find the covariance of $X$ and $Y$, if the random variable $(X,Y)$ ha[s] the joint pdf $f(x,y) = x+y$, $0 \le x \le 1$, $0 \le y \le 1$ and $f(x,y) = 0$, otherwis[e]

Solution:

Given $f(x,y) = x+y$, $0 \le x, y \le 1$.

Now, $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy = \int_0^1\int_0^1 xy(x+y)\,dx\,dy = \int_0^1\int_0^1 (x^2y + xy^2)\,dx\,[dy]$

$$= \int_0^1 (y/3 + y^2/2)\,dy = \left(\frac{y^2}{6} + \frac{y^3}{6}\right)_0^1 = \frac{1}{6} + \frac{1}{6} = 1/3.$$

Marginal density function of 'X' $f(x) = \int_0^1 f(x,y)\,dy = \int_0^1 (x+y)\,dy = \left(xy + \frac{y^2}{2}\right)_0^1$

$f(x) = x + 1/2$.

Marginal density function of 'Y' $f(y) = \int_0^1 f(x,y)\,dx = \int_0^1 (x+y)\,dx = \left(\frac{x^2}{2} + xy\right)_0^1 = y + 1/2$

Now, $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x(x+1/2)\,dx = \int_0^1 (x^2 + x/2)\,dx = \left(\frac{x^3}{3} + \frac{x^2}{4}\right)_0^1 = 7/12.$

$E(Y) = \int_{-\infty}^{\infty} y f(y)\,dy = \int_0^1 y(y+1/2)\,dy = \left(\frac{y^3}{3} + y^2/4\right)_0^1 = 7/12.$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2(x+1/2)\,dx = \int_0^1 (x^3 + x^2/2)\,dx = \left(\frac{x^4}{4} + \frac{x^3}{6}\right)_0^1 = \frac{5}{12}.$

Similarly $E(Y^2) = 5/12$.

$\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{5}{12} - (7/12)^2 = \frac{11}{144}$. $\sigma_X^2 = \frac{11}{144} \Rightarrow \sigma_X = \frac{\sqrt{11}}{12}$

Similarly $\text{Var}(Y) = \frac{11}{144}$, $\sigma_Y^2 = \frac{11}{144} \Rightarrow \sigma_Y = \frac{\sqrt{11}}{12}$.

12. b) i) The joint pdf of two dimensional random variable $(X,Y)$ is given by

$$f(x,y) = \begin{cases} \frac{8}{9}xy, & 1 < x < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

. Find the densities of $X$ and $Y$, and the conditional densities $f(x/y)$ and $f(y/x)$.

Solution:

Marginal density function of $X$ is given by $f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{2}^{x} \frac{8xy}{9}\,d$ $(x < y < 2)$.

$$f_X(x) = \frac{8}{9}x\left(\frac{y^2}{2}\right)_2^x = \frac{4x}{9}(4-x^2), \quad 1 < x < 2.$$

Similarly $f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{1}^{y} \frac{8}{9}xy\,dx = \frac{4y}{9}(y^2-1), \quad 1 < y < 2.$

The conditional density function of $Y$ given $X = x$ is

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{8/9\,xy}{\frac{4}{9}x(4-x^2)} = \frac{2y}{4-x^2}, \quad 1 < x < y < 2.$$

$$f(x/y) = \frac{f(x,y)}{f(y)} = \frac{8/9\,xy}{\frac{4}{9}y(y^2-1)} = \frac{2x}{y^2-1}, \quad 1 < x < y < 2.$$

12. b) ii) A sample of size 100 is taken from a population whose mean is 60 an[d] variance is 400. Using Central limit theorem, find the probability with which the mean of the sample will not differ from 60 by more than 4.

Solution:

To find mean $(\mu)$ and variance $(\sigma^2)$

Given $\mu = 60$; $\sigma^2 = 400$; $n = 100$ ; $\frac{\sigma}{\sqrt{n}} = \frac{20}{\sqrt{100}} = 2.$

By corollary of Lindeberg-Levy form of Central limit theorem, we have

$\bar{X} \sim N(\mu, \sigma/\sqrt{n})$ i.e. $\bar{X} \sim N(60, 2)$.

$P($ mean of the sample will not differ from $\mu = 60$ by more than $4) = P(|\bar{X}-\mu| \le$

$$= P(-4 \le \bar{X}-\mu \le 4) = P(-4 \le \bar{X}-60 \le 4) = P(60-4 \le \bar{X} \le 60+$$

$$= P(56 \le \bar{X} \le 64)$$

$$= P\left(\frac{56-60}{2} \le \frac{\bar{X}-\mu}{\sigma} \le \frac{64-60}{2}\right)$$

$$= P(-2 \le z \le 2) = P(-2 \le z \le 0) + P(0 \le z \le 2)$$

$$= 2P(0 \le z \le 2) = 2(0.4773)$$

$$\therefore P(|\bar{X}-\mu| \le 4) = 0.9546.$$

13.a) i) Examine whether the random process $\{X(t)\} = A\cos(\omega t+\theta)$ is a wide sense stationary if $A$ and $\omega$ are constants and $\theta$ is uniformly distributed random variable in $(-\pi, \pi)$.

Solution:

(i) Given $\{X(t)\} = A\cos(\omega t+\theta)$

$$\therefore E[X(t)] = \int_{-\infty}^{\infty} x(t) f(\theta) d\theta = \int_{-\pi}^{\pi} A\cos(\omega t+\theta) \cdot \frac{1}{2\pi} d\theta$$

since $\theta \sim U(-\pi, \pi)$,

$$f(\theta) = \begin{cases} \frac{1}{2\pi}, & -\pi \le \theta \le \pi \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore E[X(t)] = \frac{A}{2\pi}\left[\frac{\sin(\omega t+\theta)}{1}\right]_{-\pi}^{\pi}$$

$$= \frac{A}{2\pi}\left[\sin(\omega t+\pi) - \sin(\omega t-\pi)\right] = \frac{A}{2\pi}\left[-\sin\omega t + \sin\omega t\right] = 0$$

$$\therefore E[X(t)] = \text{constant}.$$

$\left(\begin{array}{l} \text{since } \sin(\pi+\theta) = -\sin\theta \\ \sin(\pi-\theta) = \sin\theta \\ \sin(-\theta) = -\sin\theta \end{array}\right)$

(ii) $R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$

$$= E[A\cos(\omega t+\theta) \cdot A\cos(\omega t+\omega\tau+\theta)] = A^2 E[\cos(\omega t+\theta)\cos(\omega t+\omega\tau+\theta)]$$

$\frac{1}{2}(\cos(A-B)+\cos(A+B)) = \cos A \cos B$

$$= A^2 E\left[\frac{\{\cos(-\omega\tau) + \cos(2\omega t + \omega\tau + 2\theta)\}}{2}\right]$$

$$= \frac{A^2}{2}\left[E(\cos\omega\tau) + E[\cos(2\omega t+\omega\tau+2\theta)]\right]$$

$$= \frac{A^2}{2}\left[\cos\omega\tau \int_{-\pi}^{\pi} \frac{1}{2\pi} d\theta + \int_{-\pi}^{\pi} \cos(2\omega t+\omega\tau+2\theta) \cdot \underbrace{f(\theta)}_{1/2\pi} d\theta\right]$$

$$= \frac{A^2}{2}\left[\frac{\cos\omega\tau}{2\pi}(\theta)_{-\pi}^{\pi} + \frac{1}{2\pi}\left[\frac{\sin(2\omega t+\omega\tau+2\theta)}{2}\right]_{-\pi}^{\pi}\right]$$

$$= \frac{A^2}{2}\left[\frac{\cos\omega\tau}{\cancel{2\pi}} \cdot \cancel{2\pi} + \frac{1}{4\pi}\left(\sin(2\pi+2\omega t+\omega\tau) - \sin(-2\pi+2\omega t+\omega\tau)\right)\right]$$

$$= \frac{A^2}{2}\left[\cos\omega\tau + \frac{1}{4\pi}\left(\sin(2\omega t+\omega\tau) - \sin(2\omega t+\omega\tau)\right)\right]$$

0

Since $\sin(2\pi+\theta)=\sin\theta$ ; $\sin(-\theta)=-\sin\theta$ ; $\sin(2\pi+\theta)=-\sin\theta$.

$\therefore R_{XX}(t,t+\tau)=\frac{A^2}{2}\cos\omega\tau$ = a fn of $\tau$.

Hence the Auto correlation $R_{XX}(t,t+\tau)$ depends only on time difference $\tau$ and $E[X(t)]$ = constant.

$\therefore$ $\{X(t)\}$ is wide sense stationary.

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a) ii) Assume that the number of messages input to a communication channel an interval of duration $t$ seconds, is a Poisson process with mean $\lambda=0.3$ compute 1) The probability that exactly 3 messages will arrive during 10 second interval.

2) The probability that the number of message arrivals in an interval of duration 5 seconds is between 3 and 7.

Solution:

1) Mean $\lambda=0.3$, $t=10$ sec $\therefore \lambda t=0.3\times10=3$.

Probability that exactly, 3 messages will arrive $=P(X=3)=\frac{e^{-3}(3)^3}{3!}$

$$P(X=3)=\frac{e^{-3}\times27}{6}=0.2240 \quad \text{Since } P(X=n)=\frac{e^{-\lambda t}(\lambda t)^n}{n!}.$$

2) $\lambda=0.3$ ; $t=5$ sec $\therefore \lambda t=1.5$

$$P(3<X<7)=P(X=4)+P(X=5)+P(X=6)$$

$$=\frac{e^{-1.5}(1.5)^4}{4!}+\frac{e^{-1.5}(1.5)^5}{5!}+\frac{e^{-1.5}(1.5)^6}{6!}=\frac{e^{-1.5}(1.5)^4}{4!}\left(1+\frac{1.5}{5}+\frac{(1.5)^2}{30}\right)$$

$P(3<X<7)=0.0647$.

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b) i) The random binary transmission process $\{X(t)\}$ is a WSS with zero mean and autocorrelation function $R(\tau)=1-\frac{|\tau|}{T}$, where $T$ is a constant. Find the mean and variance of the time average of $\{X(t)\}$ mean-ergodic?

Solution: Given Auto correlation function $R_{XX}(\tau)=1-\frac{|\tau|}{T}$

$E[X(t)]=0$. In $(0,T)$ the time average $\overline{X}_T=\frac{1}{T}\int_0^T X(t)dt$.

$$\therefore E[\overline{X}_T]=E\left[\frac{1}{T}\int_0^T X(t)dt\right]=\frac{1}{T}\int_0^T E[X(t)]dt=\frac{1}{T}\int_0^T 0\,dt=0$$

$Var(\overline{X_T}) = \frac{1}{2T}\int_{-2T}^{2T} R_{XX}(\tau)\, C_{XX}(\tau) d\tau$, where $C_{XX}(\tau)$ is the auto covariance function,

WKT, $C_{XX}(\tau) = R_{XX}(\tau) - E[X(t)]E[X(t+\tau)] = R_{XX}(\tau) - 0 = R_{XX}(\tau)$.

$$\therefore Var(\overline{X_T}) = \frac{1}{2T}\int_{-2T}^{2T} R_{XX}(\tau) . R_{XX}(\tau) d\tau = \frac{1}{T}\int_{-T}^{T}\left(R_{XX}(\tau)\right)^2 d\tau$$

$$= \frac{2}{T}\int_0^T\left(1 - \frac{|\tau|}{T}\right)^2 d\tau = \frac{2}{T}\int_0^T\left(1 - \frac{\tau}{T}\right)^2 d\tau = \frac{2}{T}\int_0^T\left(1 + \frac{\tau^2}{T^2} - \frac{2\tau}{T}\right) d\tau$$

$Var(\overline{X}_T) = 2/3 \quad \therefore \lim_{|\tau|\to\infty} Var(\overline{X_T}) = 2/3 \neq 0$.

$\therefore \{X(t)\}$ is not mean ergodic.

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13 b) (i) The transition probability matrix of a Markov chain $\{X_n\}$, $n = 1, 2, 3 \ldots$ having 3 states 1, 2 and 3 is $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ and the initial distribution $p^{(0)} = (0.7, 0.2, 0.1)$. Find (i) $P(X_2 = 3)$ and (ii) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$

Solution:

Given $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ $\quad P(X_0 = 1) = 0.7$, $P(X_0 = 2) = 0.2$, $P(X_0 = 3) = 0.1$

$$p^{(2)} = P^2 = P.P = \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix} \quad \text{Hence,} \quad \left.\begin{array}{l} P_{13}^{(2)} = 0.26 \\ P_{23}^{(2)} = 0.34 \\ P_{33}^{(2)} = 0.29 \end{array}\right\} \leftarrow (a)$$

To find $P(X_2 = 3)$:

$$P(X_2 = 3) = \sum_{i=1}^{3} P(X_2 = 3 / X_0 = i) . P(X_0 = i)$$

$$= P_{13}^{(2)} P(X_0 = 1) + P_{23}^{(2)} P(X_0 = 2) + P_{33}^{(2)} P(X_0 = 3).$$

$$= 0.26 \times 0.7 + 0.34 \times 0.2 + 0.29 \times 0.1 \quad \text{by (a)}$$

$$= 0.182 + 0.068 + 0.029$$

$\therefore P(X_2 = 3) = 0.279$.

To find $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$

$$P(X_1 = 3, X_0 = 2) = P(X_1 = 3 / X_0 = 2) \times P(X_0 = 2) = P_{23} \times P(X_0 = 2)$$
$$= 0.2 \times 0.2 = 0.04$$

Also $p(X_2=3, X_1=3, X_0=2) = p(X_2=3/X_1=3, X_0=2).p(X_1=3, X_0=2)$

$= p(X_2=3/X_1=3) \times p(X_1=3, X_0=2)$

$= 0.3 \times 0.04 = 0.012.$

$\therefore p(X_3=2, X_2=3, X_1=3, X_0=2) = p(X_3=2/X_2=3, X_1=3, X_0=2).p(X_2=3, X_1=3, X_0=2)$

$= p(X_3=2/X_2=3) \times p(X_2=3, X_1=3, X_0=2)$

$= 0.4 \times 0.012$

$= 0.0048$

14a) i) Find the autocorrelation function of the periodic time function $\{X(t)\} = A \sin \omega t$

$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)] = E[A\sin \omega t . A \sin(\omega t + \omega\tau)]$

$\sin A \sin B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}$

$= \frac{A^2}{2} E[\sin(2\omega t + \omega\tau) + \sin(-\omega\tau)]$

$= \frac{A^2}{2}\{E[\sin(2\omega t+\omega\tau)] + E[-\sin\omega\tau]\}$

14a) ii) The autocorrelation function of the random binary transmission $\{X(t)\}$ is given by $R_{XX}(\tau) = \begin{cases} 1 - \frac{|\tau|}{T}, & \text{for } |\tau| \le T \\ 0, & \text{for } |\tau| > T \end{cases}$ Find the power spectrum of $\{X(t)\}$

Refer to A.U. CHENNAI NOV/Dec 2010 Q.NO. 14 a) ii).

14) b) i) $\{X(t)\}$ and $\{Y(t)\}$ are zero mean and stochastically independent random processes having autocorrelation functions $R_{XX}(\tau) = e^{-|\tau|}$ and $R_{YY}(\tau) = \cos 2\pi\tau$ respectively. Find (1) The autocorrelation function of $W(t) = X(t) + Y(t)$ and $Z(t) = X(t) - Y(t)$ (2) The cross correlation fn of $W(t)$ and $Z(t)$.

Solution:

Given $\{X(t)\}$ and $\{Y(t)\}$ are random processes with zero mean

$\therefore E[X(t)] = E[Y(t)] = 0.$ —① Also $\{X(t)\}$ and $\{Y(t)\}$ are independent.

$\therefore E[XY] = E(X)E(Y)$. Given $W(t) = X(t) + Y(t)$.

Autocorrelation of $W(t) = R_{WW}(\tau) = E[W(t)W(t+\tau)]$

$= E[\{X(t)+Y(t)\}\{X(t+\tau)+Y(t+\tau)\}]$

$= E[X(t)X(t+\tau) + Y(t)X(t+\tau) + Y(t)Y(t+\tau) + Y(t+\tau)\ldots$

$= R_{XX}(\tau) + R_{YY}(\tau) + 0 + 0$

$R_{WW}(\tau) = e^{-|\tau|} + \cos 2\pi\tau$ Similarly $R_{ZZ}(\tau) = e^{-|\tau|} + \cos 2\pi\tau.$

Given $W(t) = X(t) + Y(t)$. $Z(t) = X(t) - Y(t)$.

Cross-correlation $R_{WZ}(\tau) = E[W(t) Z(t+\tau)]$

$$= E[\{X(t)+Y(t)\}\{X(t+\tau) - Y(t+\tau)\}]$$

$$= E[X(t)X(t+\tau) - X(t)Y(t+\tau) + Y(t)X(t+\tau) - Y(t)Y(t+\tau)]$$

$$= R_{XX}(\tau) - R_{YY}(\tau) - 0 + 0$$

$$R_{WZ}(\tau) = e^{-|\tau|} \cos 2\pi\tau.$$

14 b) ii) Find the autocorrelation function of the process $\{X(t)\}$ for which the pow spectral density is given by $S_{XX}(\omega) = 1+\omega^2$ for $|\omega| < 1$ and $S_{XX}(\omega) = 0$ for $|\omega| > 1$.

Solution:

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega = \frac{1}{2\pi}\int_{-1}^{1} (1+\omega^2) e^{j\omega\tau} d\omega = \frac{1}{2\pi} \times 2\int_{0}^{1} (1+\omega^2) \cos\omega\tau \, d\omega$$

$$= \frac{1}{\pi}\left[(1+\omega^2)\frac{\sin\omega\tau}{\tau} - 2\omega\left(-\frac{\cos\omega\tau}{\tau^2}\right) + 2\left(-\frac{\sin\omega\tau}{\tau^3}\right)\right]_0^1$$

$$= \frac{1}{\pi}\left[\frac{2\sin\tau}{\tau} + \frac{2\cos\tau}{\tau^2} - \frac{2\sin\tau}{\tau^3}\right]$$

$$R_{XX}(\tau) = \frac{2}{\pi\tau^3}\left[\tau^2\sin\tau + \tau\cos\tau - \sin\tau\right].$$

15a) i) A wide sense stationary random process $\{X(t)\}$ with autocorrelation $R_{XX}(\tau) = e^{-a|\tau|}$ where A and a are real positive constants, is applied to the input of an linear transmission input system with impulse response $h(t) = e^{-bt}u(t)$ where b is a real positive constant. Find the autocorrelatio of the output $Y(t)$ of the system.

Solution: Given $R_{XX}(\tau) = e^{-a|\tau|}$

The spectral density function is given by $S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} e^{-a|\tau|} e^{-j\omega\tau} d\tau = \int_{-\infty}^{0} e^{-a(-\tau)} e^{-j\omega\tau} d\tau + \int_{0}^{\infty} e^{-a\tau} e^{-j\omega\tau} d\tau$$

In $(-\infty, 0)$, $|\tau| = -\tau$, In $(0,\infty)$, $|\tau| = +\tau$.

$$S_{XX}(\omega) = \int_{-\infty}^{0} e^{a\tau} e^{-j\omega\tau} d\tau + \int_{0}^{\infty} e^{-a\tau} e^{-j\omega\tau} d\tau = \int_{-\infty}^{0} e^{(a-j\omega)\tau} d\tau + \int_{0}^{\infty} e^{-(a+j\omega)\tau} d\tau$$

$$= \frac{1}{a-i\omega} + \frac{1}{a+i\omega} = \frac{2a}{\omega^2+a^2}$$

7. Given that $h(t) = e^{-b\tau}u(t) \Rightarrow h(t) = e^{-b\tau}, \tau \geq 0$.

$$\therefore H(\omega) = \int_0^\infty e^{-b\tau} e^{-i\omega\tau} d\tau = \int_0^\infty e^{-(b+i\omega)\tau} d\tau = \frac{1}{b+i\omega}$$

$$\therefore S_{yy}(\omega) = S_{xx}(\omega) |H(\omega)|^2$$

$$= \frac{2a}{a^2+\omega^2} \left| \frac{1}{b+i\omega} \right|^2 = \frac{2a}{a^2+\omega^2} \cdot \frac{1}{b^2+\omega^2}$$

$$S_{yy}(\omega) = \frac{2a}{(a^2+\omega^2)(b^2+\omega^2)}$$

Output autocorrelation: $R_{yy}(\tau) = F^{-1}(S_{yy}(\omega))$

$$= F^{-1}\left(\frac{2a}{(a^2+\omega^2)(b^2+\omega^2)}\right)$$

Let $\frac{1}{(a^2+\omega^2)(b^2+\omega^2)} = \frac{A}{a^2+\omega^2} + \frac{B}{b^2+\omega^2}$.

$\Rightarrow 1 = A(b^2+\omega^2) + B(a^2+\omega^2)$

put $\omega^2 = -b^2$, $1 = B(a^2-b^2)$ $\therefore \boxed{B = \frac{1}{a^2-b^2} = \frac{-1}{b^2-a^2}}$

put $\omega^2 = -a^2$, $1 = A(b^2-a^2)$ $\therefore A = \frac{1}{b^2-a^2}$.

$$\therefore R_{yy}(\tau) = 2a\left[\frac{1}{b^2-a^2} F^{-1}\left(\frac{1}{a^2+\omega^2}\right) - F^{-1}\left(\frac{1}{b^2+\omega^2}\right) \cdot \frac{1}{b^2-a^2}\right]$$

$$R_{yy}(\tau) = \frac{2a}{b^2-a^2}\left[\frac{e^{-a|\tau|}}{2a} - \frac{e^{-b|\tau|}}{2b}\right] \quad \left(\text{since } F^{-1}\left[\frac{2\alpha}{\alpha^2+\omega^2}\right] = e^{-\alpha|\tau|}\right)$$

15a) (i) If $X(t)$ is a band limited process such that $S_{XX}(\omega) = 0$ when $|\omega| > \sigma$, prove that $2(R_{XX}(0) - R_{XX}(\tau)) \leq \sigma^2\tau^2 R_{XX}(0)$.

Solution:

By definition of autocorrelation,

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) \cos\omega\tau \, d\omega \quad \text{— (1)} \quad (\because \sin\omega\tau \text{ is odd})$$

and $R_{XX}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) d\omega$ — (2) (using $\tau = 0$ in (1)).

Hence,

$$R_{XX}(0) - R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) d\omega - \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) \cos\omega\tau \, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)(1-\cos\omega\tau)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{-\sigma} S_{XX}(\omega)(1-\cos\omega\tau)\,d\omega + \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)(1-\cos\omega\tau)\,d\omega + \frac{1}{2\pi}\int_{\sigma}^{\infty} S_{XX}(\omega)(1-\cos\omega\tau)$$

$$= \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)(1-\cos\omega\tau)\,d\omega \quad (\text{Since } S_{XX}(\omega)=0,\ |\omega|>\sigma).$$

$$= \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)\left(2\sin^2\left(\frac{\omega\tau}{2}\right)\right)d\omega \quad \text{—③} \quad (\text{Since } 1-\cos\theta = 2\sin^2\theta/2)$$

We know that $|\sin\theta| \le \theta \Rightarrow \sin^2\theta \le \theta^2$

$$\therefore \sin^2\left(\frac{\omega\tau}{2}\right) \le \left(\frac{\omega\tau}{2}\right)^2$$

$$\therefore R_{XX}(0) - R_{XX}(\tau) \le \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)\, 2\times\frac{\omega^2\tau^2}{4}\,d\omega$$

$$\le \frac{\omega^2\tau^2}{\cancel{8\pi}\ 4}\int_{-\sigma}^{\sigma}\cancel{2}S_{XX}(\omega)\,d\omega \quad \left(\text{Since } 2\sin^2\left(\frac{\omega\tau}{2}\right) \le 2\left(\frac{\omega\tau}{2}\right)^2 = \frac{\omega^2\tau^2}{2}.\right)$$

$$\le \frac{\omega^2\tau^2}{4\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega$$

$$\le \frac{\omega^2\tau^2}{2}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega\right)$$

$$\le \frac{\omega^2\tau^2}{2} R_{XX}(0), \quad \omega < \sigma$$

$$\therefore 2\left(R_{XX}(0) - R_{XX}(\tau)\right) \le \omega^2\tau^2 R_{XX}(0), \quad \omega < \sigma$$

$$\le \sigma^2\tau^2 R_{XX}(0)$$

15 b) i) Assume a random process $X(t)$ is given as input to a system with transfer function $H(\omega)=1$ for $-\omega_0 < \omega < \omega_0$. If the auto correlation function of the input process is $\frac{N_0}{2}\delta(\tau)$, find the auto correlation function of the output proce

Given $R_{XX}(\tau) = \frac{N_0}{2}\delta(\tau)$, where $R_{XX}(\tau)$ is the input autocorrelation function. Taking Fourier transform on both sides, $F[R_{XX}(\tau)] = \frac{N_0}{2}F[\delta(\tau)]$.

$S_{XX}(\omega) = \frac{N_0}{2}$

But $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

$= 1^2 \times \frac{N_0}{2}, \quad -\omega_0 < \omega < \omega_0$

$\therefore$ The output autocorrelation function is given by,

$R_{YY}(\tau) =$ Inverse Fourier transform of $S_{YY}(\omega)$.

$$\therefore R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)\, e^{j\omega\tau} d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} \frac{N_0}{2} e^{j\omega\tau} d\omega = \frac{N_0}{4\pi}\left[\frac{e^{j\omega\tau}}{j\tau}\right]_{-\omega_0}^{\omega_0}$$

$$= \frac{N_0}{4\pi}\left[\frac{e^{j\omega_0\tau} - e^{-j\omega_0\tau}}{j\tau}\right] = \frac{N_0}{4\pi\tau}\, 2\sin\omega_0\tau$$

$$R_{YY}(\tau) = \frac{N_0 \sin\omega_0\tau}{2\pi\tau}$$

15 b) ii) Let $Y(t) = A\cos(\omega_0 t + \theta) + N(t)$ where $A$ is a constant, $\theta$ is a RV with a uniform distribution in $(-\pi, \pi)$ and $\{N(t)\}$ is band limited Gaussian white noise with a power spectral density. $S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}, & |\omega - \omega_0| \le \omega_B \\ 0, & \text{otherwise.} \end{cases}$

Find the spectral density function of $\{Y(t)\}$ assuming that $N(t)$ and $\theta$ are independent.

Refer A.U Chennai Nov.Dec 2010 Q.No. 15b) ii).