B.E / B.Tech Degree Examination; April / May 2008

MA 034 – Random Processes

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Answer ALL questions

Part A – (10 x 2 = 20 marks)

1. The odds in favour of A solving a mathematical problem are 3 to 4 and the odds against B solving the problems are 5 to 7. Find the probability that the problem will be solved by atleast one of them.

Out of Syllabus.

2. A and B are events such that $P(A) = 3/8$, $P(B) = 1/2$ and $P(A \cap B) = 1/4$. Find $P(\bar{A} \cap \bar{B})$. Out of syllabus.

3. A random variable X has the pdf f(x) given by $f(x) = \begin{cases} Cxe^{-x}, & x > 0 \\ 0, & x \le 0 \end{cases}$ Find the value of C and cdf of X.

Since f(x) is a pdf, $\int_{-\infty}^{\infty} f(x)dx = 1$

i.e $\int_0^{\infty} Cxe^{-x}dx = 1$ i.e $C\left[x\left(\frac{e^{-x}}{-1}\right) - (1)\left(\frac{e^{-x}}{+1}\right)\right]_0^{\infty} = 1 \Rightarrow C\left[\{0-0\} + \{0+ \right.$

$= 1$

$\therefore C(1) = 1 \Rightarrow \boxed{C = 1}$

$\therefore$ pdf $f(x) = \begin{cases} xe^{-x}, & x > 0 \\ 0, & x \le 0. \end{cases}$

cdf of X, $F(x) = P(X \le x)$

$F(x) = 0$, when $x < 0$.

$F(x) = \int_0^x xe^{-x}dx = \left[x\left(\frac{e^{-x}}{-1}\right) - (1)\left(\frac{e^{-x}}{(-1)^2}\right)\right]_0^x = \left[\{-xe^{-x} - e^{-x}\} - \{0 - 1\}\right]$

$= -xe^{-x} - e^{-x} + 1, \quad x > 0$

$\therefore F(x) = \begin{cases} 0, & x \le 0 \\ 1 - e^{-x} - xe^{-x}, & x > 0. \end{cases}$

4. If X has an exponential distribution with parameter α, find the p.d.f of $Y = \log X$.

Since X follows exponential distribution with parameter a

pdf of X is $f(x) = a e^{-ax}$, $a > 0$, $0 < x < \infty$.

Given: $y = \log x \Rightarrow x = e^y$

$$\frac{dx}{dy} = e^y \Rightarrow \left|\frac{dx}{dy}\right| = e^y.$$

$\therefore$ pdf of Y is $f(y) = f(x) \left|\frac{dx}{dy}\right|$

$$= a e^{-ax} e^y$$

$$f(y) = a e^{-ae^y} \cdot e^y \quad , \quad 1 < y < \infty.$$

5. If X_1 has mean 4 and Variance 9 while X_2 has mean -2 and Varia[nce 5 and] the two are independent, find $Var(2X_1 + X_2 - 5)$.

Given X_1 and X_2 are independent, $Var(2X_1 + X_2 - 5) = Var(2X_1) + Var(X_2 - 5)$

$\therefore Var(2X_1 + X_2 - 5) = Var(2X_1) + Var(X_2 - 5)$

$$= 2^2 Var(X_1) + Var(X_2) \quad \left(\because Var(aX+b) = a^2 Var(X); \ Var(b) = 0.\right)$$

$$= 4 Var(X_1) + Var(X_2)$$

$$= 4(9) + 5 = 36 + 5$$

$$Var(2X_1 + X_2 - 5) = 41$$

6. State four types of stochastic processes.

1. Discrete Random Sequence
2. Discrete Random process
3. Continuous Random Sequence
4. Continuous Random Process.

7. Define a Poisson variate

If $X(t)$ represents the number of occurrences of a certain event in $(0, t)$, then the discrete random process $\{X(t)\}$ is called the poisson process, provided the following postulates are satisfied.

(i) $P[1 \text{ occurrence in } (t, t+\Delta t)] = \lambda \Delta t + O(\Delta t)$

(ii) $P[0 \text{ occurrence in } (t, t+\Delta t)] = 1 - \lambda \Delta t + O(\Delta t)$

(iii) $P[2 \text{ occurrence or more occurrences in } (t, t+\Delta t)] = O(\Delta t)$

(iv) $X(t)$ is independent of the number of occurrences of the event in any interval prior (or) after the interval $(0, t)$.

The probability that the event occurs a specified number of times in (t_0, t_0+t) depends only on t, but not on t_0.

8. Prove that a first order stationary random process has a constant mean.

If $\{X(t)\}$ is first order stationary, we have $f(x,t) = f(x, t+\Delta), \forall t \& \Delta$.

$\Rightarrow f(x,t)$ is independent of time t

$\Rightarrow E[X(t)] = \int_{-\infty}^{\infty} x\, f(x,t)\, dx$ is independent of time.

i.e. $E[X(t)] = \text{constant } (\mu) = \text{mean}$.

and $\therefore$ If mean $\neq$ constant, then the process is not first order stationary.

9. Define autocorrelation function.

The autocorrelation function $R_{XX}(\tau)$ is defined by $R_{XX}(\tau) = E[X(t)X(t+\tau)]$

10. If the autocorrelation function of a stationary process is $R_{XX}(\tau) = 36 + \frac{4}{1+3\tau^2}$, find the mean and variance of the process.

Given $R_{XX}(\tau) = 36 + \frac{4}{1+3\tau^2}$

By the property of ACF, $\bar{X}^2 = \lim_{|\tau|\to\infty} R_{XX}(\tau)$

$$\therefore \bar{X}^2 = \lim_{|\tau|\to\infty} \left[36 + \frac{4}{1+3\tau^2}\right]$$

$$\bar{X}^2 = 36 \quad \Rightarrow \bar{X} = E[X(t)] = \text{mean} = \sqrt{36} = 6.$$

We know that, $E[X^2(t)] = R_{XX}(0) = 36 + \frac{4}{1+0} = 36 + 4 = 40.$

$\therefore$ Variance of $\{X(t)\} = E[X^2(t)] - \{E[X(t)]\}^2 = 40 - 6^2 = 40 - 36 = 4$.

$\therefore$ Mean $= 6$ and Variance $= 4$.

PART - B (5 × 16 = 80 marks)

11 a) (i) In a certain group of engineers 60% have insufficient background of information theory. 50% have inadequate knowledge of probability and 80% are in either one or both of two categories. What is percentage of people

who have adequate knowledge of probability among those who hav[e] sufficient background of information theory?

Out of syllabus.

ii) Find the moment generating function of the random variable a[s] the probability law $p(X=x) = q^{x-1}p$, $x = 1,2,3 \dots$ Also find the mea[n] and variance.

The moment generating function is given by $M_X(t) = E(e^{tX})$

$$= \sum_{x=0}^{\infty} e^{tx} p(X=x) = \sum_{x=1}^{\infty} e^{tx} q^{x-1} p = q^{-1} p \sum_{x=1}^{\infty} e^{tx} q^{x} = \frac{p}{q} \sum_{x=1}^{\infty} (qe^{t})^{x}$$

$$= \frac{p}{q}\left[qe^{t} + (qe^{t})^2 + (qe^{t})^3 + \dots\right]$$

$$= \frac{p}{\cancel{q}} \cancel{q} e^{t}\left[1 + qe^{t} + (qe^{t})^2 + \dots\right] = pe^{t}(1-qe^{t})^{-1} \quad (\because (1-x)^{-1} = 1+x+$$

$$M_X(t) = \frac{pe^{t}}{1-qe^{t}} \; ; \; M_X'(t) = \frac{d}{dt}\left(\frac{pe^{t}}{1-qe^{t}}\right) = p\left[\frac{(1-qe^{t})e^{t} - e^{t}(-qe^{t})}{(1-qe^{t})^2}\right] = \frac{p[e^t]}{(1-q[e^t)^2]}$$

$$E[X] = M_X'(0) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = 1/p.$$

$$M_X''(t) = \frac{d^2}{dt^2}(M_X(t)) = \frac{d}{dt}\left(\frac{pe^{t}}{(1-qe^{t})^2}\right) = p\left[\frac{(1-qe^{t})^2 e^{t} - e^{t}\,2(1-qe^{t})(-qe^{t})}{(1-qe^{t})^4}\right]$$

$$E[X^2] = M_X''(0) = p\left[\frac{(1-q)^2 - 2(1-q)(-q)}{(1-q)^4}\right] = p\left[\frac{p^2+2pq}{p^4}\right] = \frac{\cancel{p^2}(p+2q)}{\cancel{p^4}\,p^2}$$

$$= \frac{p+q+q}{p^2} = \frac{1+q}{p^2} = \frac{1}{p^2} + \frac{q}{p^2}$$

$$\therefore Var(X) = E(X^2) - (E(X))^2 = \cancel{\frac{1}{p^2}} + \frac{q}{p^2} - \cancel{(1/p)^2} = \frac{q}{p^2} \,//.$$

b) i) Suppose the duration 'x' in minutes of long distance calls from your home, follows exponential law with pdf $f(x) = \frac{1}{5}e^{-x/5}$ for $x > 0$, 0 otherwise. Find $P(X>5)$, $p(3<X<6)$, mean and variance [of] X.

Here $\lambda = \frac{1}{5}$

(i) $$P(X>5) = \int_5^{\infty} f(x)dx = \int_5^{\infty} \frac{1}{5} e^{-x/5} dx = \frac{1}{5}\left[\frac{e^{-x/5}}{-1/5}\right]_5^{\infty} = -\left[e^{-\infty} - e^{-1}\right] = e^{-1}.$$

(i) $P(3 \le X \le 6) = \int_3^6 \frac{1}{5} e^{-x/5} dx = \cancel{\frac{1}{5}} \left[\frac{e^{-x/5}}{\cancel{-1/5}} \right]_3^6 = -\left[e^{-6/5} - e^{-3/5} \right] = e^{-3/5} - e^{-6/5}.$ (3)

(ii) Mean $= \frac{1}{\lambda} = \frac{1}{1/5} = 5$

Variance $= \frac{1}{\lambda^2} = \frac{1}{1/25} = 1 \times \frac{25}{1} = 25.$

ii) Suppose that the lifetime of a certain kind of an emergency backup battery (in hours) is a random variable X having the Weibull distribution with $\alpha = 0.1$ and $\beta = 0.5$ find (1) the mean life time of these batteries
(2) the probability that such a battery will last more than 300 hours
(3) the probability that such a battery will not last 100 hours.

Out of syllabus.

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15 a) i) The joint probability mass function of X and Y is

X \ Y	0	1	2
0	0.1	0.04	0.02
1	0.08	0.20	0.06
2	0.06	0.14	0.30

Compute the marginal distribution function of X and Y, $P(X \le 1, Y \le 1)$ and check whether X and Y are independent.

Solution:

The marginal distributions are given in the table below:

X \ Y	0	1	2	$P(X=x)$
0	0.10 $P(0,0)$	0.04 $P(0,1)$	0.02 $P(0,2)$	0.16 $P(X=0)$
1	0.08 $P(1,0)$	0.20 $P(1,1)$	0.06 $P(1,2)$	0.34 $P(X=1)$
2	0.06 $P(2,0)$	0.14 $P(2,1)$	0.30 $P(2,2)$	0.50 $P(X=2)$
$P(Y=y)$	$P(Y=0)=0.24$	$P(Y=1)=0.38$	$P(Y=2)=0.38$	1

The Marginal distribution of X

$P(X=0) = P(0,0) + P(0,1) + P(0,2) = 0.10 + 0.04 + 0.02 = 0.16$

$P(X=1) = P(1,0) + P(1,1) + P(1,2) = 0.08 + 0.20 + 0.06 = 0.34$

$P(X=2) = P(2,0) + P(2,1) + P(2,2) = 0.06 + 0.14 + 0.30 = 0.50$

The Marginal distribution of Y

$P(Y=0) = P(0,0) + P(1,0) + P(2,0) = 0.10 + 0.08 + 0.06 = 0.24$

$P(Y=1) = P(0,1) + P(1,1) + P(2,1) = 0.04 + 0.20 + 0.14 = 0.38$

$P(Y=2) = P(0,2) + P(1,2) + P(2,2) = 0.02 + 0.06 + 0.30 = 0.38$

$P(X \le 1, Y \le 1) = P(0,0) + P(1,0) + P(0,1) + P(1,1) = 0.1 + 0.08 + 0.04 + 0.2 = 0.42$

To test X and Y are independent

$P(X=0)\,P(Y=0) = 0.16 \times 0.24 \neq 0.1$

i.e $P(X=0)P(Y=0) = P(X=0, Y=0)$. $\therefore$ X and Y are not independent.

ii) If the independent random variables X and Y have the variance 36 and 16 respectively, find the correlation coefficient between $(X+Y)$ and $(X-Y)$

Given that $Var(X) = 36$, $Var(Y) = 16$. Since X and Y are independent,

$E[XY] = E(X).E(Y)$. Let $U = X+Y$, $V = X-Y$

$Var(U) = Var(X+Y) = 1^2 Var(X) + 1^2 Var(Y) = 36 + 16 = 52$ ($\because Var(X) = 36, Var(Y) = 16$)

$Var(V) = Var(X-Y) = 1^2 Var(X) + (-1)^2 Var(Y) = Var(X) + Var(Y) = 36 + 16 = 52$

$\therefore \sigma_U = \sqrt{52} \Rightarrow \sigma_V = \sqrt{52}$.

$Cov(U,V) = E(UV) - E(U)E(V)$

$= E(X^2) - E(Y^2) - \big(E(X) + E(Y)\big)\big(E(X) - E(Y)\big)$

$= E(X^2) - E(Y^2) + (E(X))^2$
$+ (E(Y))^2 - E(X)E(Y) + E(X)E(Y)$

(Since $E(UV) = E[(X+Y)(X-Y)]$
$= E[X^2 - \cancel{XY} + Y^2 + \cancel{XY}]$
$= E(X^2) - E(Y^2)$

$E(U) = E(X+Y) = E(X) + E(Y)$

$E(V) = E(X-Y) = E(X) - E(Y)$)

$= \{E(X^2) - (E(X))^2\}\{E(Y^2) - (E(Y))^2\}$

$= Var(X) - Var(Y)$

$= 36 - 16$

$= 20$

Hence $\rho(U,V) = \dfrac{Cov(U,V)}{\sigma_U \sigma_V} = \dfrac{20}{\sqrt{52}\sqrt{52}} = \dfrac{20}{52} = \dfrac{5}{13}$.

12) b) (i) The joint pdf of X and Y is given by $f(x,y) = \begin{cases} kx(x-y), & 0<x<2, \ -x<y<x \\ 0, & \text{otherwise} \end{cases}$

evaluate the constant k and find the marginal probability density functions of the random variables. Find also conditional of Y given X=x.

(i) WKT, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1.$

Here $f(x,y) = kx(x-y),\ 0<x<2,\ -x<y<x.$

$\therefore \int_0^2 \int_{-x}^{x} kx(x-y)\,dy\,dx = 1.$

$k\int_0^2 \left[x^2y - x\frac{y^2}{2}\right]_{-x}^{x} dx = 1 \Rightarrow k\int_0^2 \left[\left(x^3 - \frac{x^3}{2}\right) - \left(-x^3 - \frac{x^3}{2}\right)\right]dx = 1.$

$\Rightarrow k\int_0^2 (2x^3)\,dx = 1 \Rightarrow 2k\int_0^2 x^3\,dx = 1 \Rightarrow 2k\left[\frac{x^4}{4}\right]_0^2 = 1 \Rightarrow 2k\left(\frac{16}{4} - 0\right) = 1.$

$\boxed{\therefore c = 1/8}$

$\therefore$ pdf is $f(x,y) = \frac{1}{8}x(x-y),\ 0<x<2,\ -x<y<2.$

(ii) $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy.$

Here $f(x) = \int_{-x}^{x} \frac{1}{8}x(x-y)\,dy = \frac{1}{8}\int_{-x}^{x}(x^2 - xy)\,dy = \frac{1}{8}\left[x^2y - \frac{xy^2}{2}\right]_{-x}^{x}$

$= \frac{1}{8}\left[\left(x^3 - \frac{x^3}{2}\right) - \left(-x^3 - \frac{x^3}{2}\right)\right]$

$= \frac{1}{8} \times 2x^3 = \frac{x^3}{4}$

$\therefore f(x) = \frac{x^3}{4},\ 0<x<2.$

(iii) $f(y/x) = \frac{f(x,y)}{f(x)} = \frac{\frac{1}{8}x(x-y)}{x^3/4} = \frac{1}{2x^2}(x-y),\ -x<y<2$

(ii) Calculate the Karl-Pearson's coefficient of correlation from the following data:

X:	39	65	62	90	82	75	25	98	36	78
Y:	47	53	58	86	62	68	60	91	51	84

X	39	65	62	90	82	75	25	98	36	78
Y	47	53	58	86	62	68	60	91	51	84
XY	1833	3445	3596	7740	5084	5100	1500	8918	1836	655
X^2	1521	4225	3844	8100	6724	5625	625	9604	1296	608
Y^2	2209	2809	3364	7396	3844	4624	3600	8281	2601	705

$\bar{X} = \frac{\Sigma X}{n} = \frac{650}{10} = 65$; $\Sigma X^2 = 47,648$; $\Sigma XY = 45,604$

$\bar{Y} = \frac{\Sigma Y}{n} = \frac{660}{10} = 66$; $\Sigma Y^2 = 45,784$. $\frac{\Sigma XY}{10} = 4560.4$

$\bar{X}.\bar{Y} = 65 \times 66 = 4290.$

$$\sigma_x = \sqrt{\frac{1}{n}\Sigma x^2 - \bar{x}^2} = \sqrt{4764.8 - 4225} = 23.233.$$

$$\sigma_y = \sqrt{\frac{1}{n}\Sigma y^2 - \bar{y}^2} = \sqrt{4578.4 - 4356} = 14.91.$$

$$\therefore r(x,y) = \frac{Cov(x,y)}{\sigma_x \sigma_y} = \frac{\frac{1}{n}\Sigma xy - \bar{x}.\bar{y}}{\sigma_x \sigma_y} = \frac{4560.4 - 4290}{23.23 \times 14.91} = \frac{270.4}{23.23 \times 14.91}$$

$\therefore r(x,y) = 0.7807$.

13) a) i) If the process $\{N(t);\ t \geq 0\}$ is a Poisson process with parameter λ, obtain $P[N(t) = n]$ and $E[N(t)]$.

Let λ be the mean rate of occurrence of the event in unit time i.e, the mean number of occurrences of the event in unit time and let $P_n(t)$ represent the probability of n occurrences of the event in the interval $(0,t)$.

i.e $P_n(t) = P(X(t) = n)$.

$$P_n(t+\Delta t) = P[X(t+\Delta t) = n]$$

$$= P_n(t) - P_n(t)(\lambda \Delta t) + P_{n-1}(t)(\lambda \Delta t)$$

$$P_n(t+\Delta t) - P_n(t) = (\Delta t)\lambda\,[P_{n-1}(t) - P_n(t)]$$

$$\therefore \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \lambda\,[P_{n-1}(t) - P_n(t)]$$

$\therefore$ Taking limits as $\Delta t \to 0$ we get,

$$\lim_{\Delta t\to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \lambda\,[P_{n-1}(t) - P_n(t)].$$

$$\frac{d}{dt}P_n(t) = \lambda P_{n-1}(t) - \lambda P_n(t).$$

ie $\frac{d}{dt}P_n(t) + \lambda P_n(t) = \lambda P_{n-1}(t)$ — ① which is a linear differential equation. $\therefore$ The solution of ① is,

$$P_n(t)\, e^{\lambda t} = \int_0^t \lambda P_{n-1}(t)\, e^{\lambda t}\, dt \quad \text{— ②}.$$

Taking $n=1$ we get,

$$e^{\lambda t} P_1(t) = \lambda \int_0^t P_0(t)\, e^{\lambda t}\, dt \quad \text{— ③}$$

$P_0(t+\Delta t) = P_0(t) - P_0(t)(\lambda\Delta t)$ $\therefore \frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t)$.

$$\Rightarrow \lim_{\Delta t\to 0} \frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t)$$

ie $\frac{d}{dt}P_0(t) = -\lambda P_0(t)$. $\Rightarrow \frac{dP_0(t)}{P_0(t)} = -\lambda\, dt$.

Integrating, $\log P_0(t) = -\lambda t + c$ $\therefore \boxed{P_0(t) = A e^{-\lambda t}}$ where $A = e^c$

Using $t=0$, $P_0(0) = Ae^0 = A$ ie $\boxed{A=1}$ $\therefore P_0(t) = e^{-\lambda t}$

$$\therefore ③ \Rightarrow e^{\lambda t} P_1(t) = \lambda \int_0^t e^{-\lambda t} e^{\lambda t}\, dt$$

$$= \lambda \int_0^t dt = \lambda t$$

$$\therefore P_1(t) = e^{-\lambda t}\lambda(t)$$

Similarly, Putting $n=2$ in (2),

$$P_2(t)e^{\lambda t} = \lambda\int_0^t P_1(t)e^{\lambda t}dt$$

$$= \lambda\int_0^t e^{-\lambda t}(\lambda t)e^{\lambda t}dt$$

$$= \lambda^2\int_0^t t\,dt = \lambda^2\left(\frac{t^2}{2}\right)_0^t$$

$$= \frac{\lambda^2 t^2}{2}$$

$$\therefore P_2(t) = \frac{e^{-\lambda t}(\lambda t)^2}{2!}$$

In general $P[N(t) \text{ or } X(t) = n] = P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$, $n=0,1,2,\ldots$

Mean $= E[X(t) \text{ or } N(t)] = \sum_{x=0}^{\infty} n\,P(N(t)=n)$

$$= \sum_{x=0 \text{ or } n}^{\infty} n \cdot \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= e^{-\lambda t}\sum_{x=1 \text{ or } n}^{\infty}\frac{(\lambda t)^n}{(n-1)!} = e^{-\lambda t}\left[\lambda t + \frac{(\lambda t)^2}{1!} + \frac{(\lambda t)^3}{2!} + \cdots\right]$$

$$= e^{-\lambda t}(\lambda t)\left[1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots\right]$$

$$= e^{-\lambda t}(\lambda t)^n \cdot e^{\lambda t}$$

$\therefore$ Mean $= E[N(t)] = \lambda t$

ii) A random sample of size 100 is taken from a population whose mean 60 and variance 400. Using CLT with what probability can we assert that the mean of the sample will not differ from $\mu=60$ more than 4?

Given $\mu=60$, $\sigma^2=400$, $n=100$

Refer A.U. Chennai April/May 2010 Qn. 12 b ii).

13b) i) Show that the process $X(t) = A\cos\lambda t + B\sin\lambda t$ is weak sense stationary? where A and B are random variables with $E(A)=E(B)=0$, $E(B^2)=E(A^2)$ and $E(AB)=0$.

(i) $E[X(t)] = E[A\cos\lambda t] + E[B\sin\lambda t] = \cos\lambda t\, E(A) + \sin\lambda t\, E(B)$

$= \cos\lambda t\,(0) + \sin\lambda t\,(0)$ (since $E(A)=E(B)=0$)

$= 0$

ii) $R(t_1, t_2) = E[X(t_1)X(t_2)] = E[(A\cos\lambda t_1 + B\sin\lambda t_1)(A\cos\lambda t_2 + B\sin\lambda t_2)]$

$= E[A^2\cos\lambda t_1\cos\lambda t_2 + AB\sin\lambda t_1\cos\lambda t_2 + AB\cos\lambda t_1\sin\lambda t_2 + B^2\sin\lambda t_1\sin\lambda t_2)$

$= \cos\lambda t_1\cos\lambda t_2\, E(A^2) + \sin\lambda t_1\cos\lambda t_2\, E\cancel{(AB)}^{0} + \cos\lambda t_1\sin\lambda t_2\, E\cancel{(BA)}^{0}$
$+ \sin\lambda t_1\sin\lambda t_2\, E(B^2)$

$= E(A^2)\{\cos\lambda t_1\cos\lambda t_2 + \sin\lambda t_1\sin\lambda t_2\}$ (since $E(AB)=0$ and $E(A^2)=E(B^2)$)

$= E(A^2)\cos\lambda(t_1 - t_2)$

$R(t_1,t_2)$ = a function of time difference

$\therefore$ From ① & ②, $X(t) = A\cos\lambda t + B\sin\lambda t$ is a WSS process.

b) ii) A random process be given as $Z(t) = X(t)\cos(\omega_0 t + \theta)$ where $X(t)$ is stationary random process with $E(X(t))=0$ and $E(X^2(t)) = \sigma_x^2$. If θ is a R.V independent of $X(t)$ and uniformly distributed over the interval $(-\pi,\pi)$. Show that $E[Z(t)]=0$ and $E[Z^2(t)] = \sigma_x^2/2$.

Given $X(t)$ and θ are independent RV and process, $\theta \sim U(-\pi,\pi)$

$\therefore f(\theta) = \begin{cases} \frac{1}{2\pi}, & (-\pi,\pi) \\ 0, & \text{otherwise} \end{cases}$

(i) $E[Z(t)] = E[X(t)\cos(\omega_0 t+\theta)] = E[X(t)]\cdot E[\cos(\omega_0 t+\theta)] = 0.$

(ii) $E[Z^2(t)] = E[X^2(t)\cos^2(\omega_0 t+\theta)] = E[X^2(t)]\, E[\cos^2(\omega_0 t+\theta)] = \sigma_x^2 E[\cos^2(\omega_0 t+$

$= \sigma_x^2 E\left[\frac{1+\cos 2(\omega_0 t+\theta)}{2}\right] = \sigma_x^2 E(1/2) + \sigma_x^2 E\left[\frac{\cos(2\omega_0 t+2\theta)}{2}\right]$

$= \frac{\sigma_x^2}{2} + \int_{-\pi}^{\pi}\frac{\cos(2\omega_0 t+2\theta)}{2}\cdot f(\theta)\,d\theta = \frac{\sigma_x^2}{2} + \int_{-\pi}^{\pi}\frac{\cos(2\omega_0 t+2\theta)}{2}\frac{1}{2\pi}d\theta$

$E(Z^2(t)) = \frac{\sigma_x^2}{2}$ // since $\left[\sin(2\omega_0 t+2\theta)\right]_{-\pi}^{\pi} = 0.$

14a) i) Prove that the power spectrum of the time function $e^{-\alpha|t|}(1+\frac{\alpha}{t})$ is $\frac{4a^3}{}$ or $e^{-\alpha|\tau|}(1+\alpha|\tau|)$ $(a^2+\omega^2)^2$

Given $R_{XX}(\tau) = e^{-\alpha|\tau|}(1+\alpha|\tau|)$

The power spectrum $S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} e^{-\alpha|\tau|}(1+\alpha|\tau|) e^{-j\omega\tau} d\tau$

$$= \int_{-\infty}^{0} e^{-\alpha|\tau|}(1+\alpha|\tau|) e^{-j\omega\tau} d\tau + \int_{0}^{\infty} e^{-\alpha|\tau|}(1+\alpha|\tau|) e^{-j\omega\tau} d\tau \quad \text{--- (1)}$$

$$= \int_{-\infty}^{0} e^{\alpha\tau}(1-\alpha\tau) e^{-j\omega\tau} d\tau + \int_{0}^{\infty} e^{-\alpha\tau}(1+\alpha\tau) e^{-j\omega\tau} d\tau \quad \text{since } |\tau| = \begin{cases} \tau, & \tau \geq 0 \\ -\tau, & \tau < 0 \end{cases}$$

$$= \int_{-\infty}^{0} e^{(\alpha-j\omega)\tau}(1-\alpha\tau) d\tau + \int_{0}^{\infty} e^{-(\alpha+j\omega)\tau}(1+\alpha\tau) d\tau$$

$$= \left[(1-\alpha\tau)\frac{e^{(\alpha-j\omega)\tau}}{\alpha-j\omega} - (-\alpha)\frac{e^{(\alpha-j\omega)\tau}}{(\alpha-j\omega)^2}\right]_{-\infty}^{0} + \left[(1+\alpha\tau)\frac{e^{-(\alpha+j\omega)\tau}}{-(\alpha+j\omega)} + (-\alpha)\frac{e^{-(\alpha+j\omega)\tau}}{(\alpha+j\omega)^2}\right]_{0}^{\infty}$$

$$= \frac{1}{\alpha-j\omega} + \frac{1}{\alpha+j\omega} + \alpha\left(\frac{(\alpha+j\omega)^2 + (\alpha-j\omega)^2}{(\alpha^2+\omega^2)^2}\right) \quad (\because e^{-\infty} = 0, e^{0} = 1)$$

$$= \frac{2\alpha}{\alpha^2+\omega^2} + \frac{2\alpha(\alpha^2-\omega^2)}{(\alpha^2+\omega^2)^2}$$

$$= \frac{2\alpha(\alpha^2+\omega^2) + 2\alpha(\alpha^2-\omega^2)}{(\alpha^2+\omega^2)^2} = \frac{2\alpha^3 + \cancel{2\alpha\omega^2} + 2\alpha^3 - \cancel{2\alpha\omega^2}}{(\alpha^2+\omega^2)^2}$$

$$S_{XX}(\omega) = \frac{4\alpha^3}{(\alpha^2+\omega^2)^2}.$$

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ii) The ACF of the random process $X(t)$ is given by $R(\tau) = \begin{cases} 1-\frac{|\tau|}{T}, & |\tau| \leq T \\ 0, & |\tau| > T \end{cases}$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{-T} R_{XX}(\tau) e^{-j\omega\tau} d\tau + \int_{-T}^{T} R_{XX}(\tau) e^{-j\omega\tau} d\tau + \int_{T}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau.$$

Given

$$R_{XX}(\tau) = \begin{cases} 1-\frac{|\tau|}{T}, & -T \leq \tau \leq T \\ 0, & \text{Otherwise}. \end{cases}$$

13b) i) Show that the process $X(t) = A\cos\lambda t + B\sin\lambda t$ is weak Sense stationary? where A and B are random variables with $E(A) = E(B) = 0$, $E(B^2) = E(A^2)$ and $E(AB) = 0$.

(i) $E[X(t)] = E[A\cos\lambda t] + E[B\sin\lambda t] = \cos\lambda t\, E(A) + \sin\lambda t\, E(B)$

$= \cos\lambda t\,(0) + \sin\lambda t\,(0)$ $\left(\text{since } E(A) = E(B) = 0\right)$

$= 0$

ii) $R(t_1, t_2) = E[X(t_1)X(t_2)] = E[(A\cos\lambda t_1 + B\sin\lambda t_1)(A\cos\lambda t_2 + B\sin\lambda t_2)]$

$= E[A^2\cos\lambda t_1\cos\lambda t_2 + AB\sin\lambda t_1\cos\lambda t_2 + AB\cos\lambda t_1\sin\lambda t_2 + B^2\sin\lambda t_1\sin\lambda t_2]$

$= \cos\lambda t_1\cos\lambda t_2\, E(A^2) + \sin\lambda t_1\cos\lambda t_2\, E(\cancel{AB})^{0} + \cos\lambda t_1\sin\lambda t_2\, E(\cancel{BA})^{0}$
$+ \sin\lambda t_1 \sin\lambda t_2\, E(B^2)$

$= E(A^2)\{\cos\lambda t_1\cos\lambda t_2 + \sin\lambda t_1\sin\lambda t_2\}$ $\left(\text{since } E(AB) = 0 \text{ and } E(A^2) = E(B^2)\right)$

$= E(A^2)\cos\lambda(t_1 - t_2)$

$R(t_1, t_2) =$ a function of time difference

$\therefore$ From ① & ②, $X(t) = A\cos\lambda t + B\sin\lambda t$ is a WSS process.

b) ii) A random process be given as $Z(t) = X(t)\cos(\omega_0 t + \theta)$ where $X(t)$ is stationary random process with $E(X(t)) = 0$ and $E(X^2(t)) = \sigma_X^2$. If θ is a R.V independent of $X(t)$ and uniformly distributed over the interval $(-\pi, \pi)$. Show that $E[Z(t)] = 0$ and $E[Z^2(t)] = \sigma_X^2/2$.

Given $X(t)$ and θ are independent RV and process, $\theta \sim U(-\pi, \pi)$

$\therefore f(\theta) = \begin{cases} \frac{1}{2\pi}, & (-\pi, \pi) \\ 0, & \text{otherwise} \end{cases}$

(i) $E[Z(t)] = E[X(t)\cos(\omega_0 t + \theta)] = E[X(t)] \cdot E[\cos(\omega_0 t + \theta)] = 0.$

(ii) $E[Z^2(t)] = E[X^2(t)\cos^2(\omega_0 t + \theta)] = E[X^2(t)]\,E[\cos^2(\omega_0 t + \theta)] = \sigma_X^2\, E[\cos^2(\omega_0 t + \theta)]$

$= \sigma_X^2\, E\left[\frac{1 + \cos 2(\omega_0 t + \theta)}{2}\right] = \sigma_X^2\, E(1/2) + \sigma_X^2\, E\left[\frac{\cos(2\omega_0 t + 2\theta)}{2}\right]$

$= \frac{\sigma_X^2}{2} + \int_{-\pi}^{\pi} \frac{\cos(2\omega_0 t + 2\theta)}{2} \cdot f(\theta)\,d\theta = \frac{\sigma_X^2}{2} + \int_{-\pi}^{\pi} \frac{\cos(2\omega_0 t + 2\theta)}{2}\, \frac{1}{2\pi}\,d\theta$

$E(Z^2(t)) = \frac{\sigma_X^2}{2}$ // since $\left[\sin(2\omega_0 t + 2\theta)\right]_{-\pi}^{\pi} = 0$.

(7)

$\therefore S_{XX}(\omega) = \int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right)e^{-j\omega\tau}d\tau = \int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right)\cos\omega\tau\, d\tau \quad \left(\because \int_{-T}^{T}\underbrace{\left(1-\frac{|\tau|}{T}\right)\sin\omega\tau\, d\tau}_{odd\,=0}\right)$

$= \int_{-T}^{T} = 2\int_{0}^{T}\left(1-\frac{\tau}{T}\right)\cos\omega\tau\, d\tau \quad \left(\because |\tau| = +\tau \text{ in } (0,T)\right)$

$= 2\left[\left(1-\frac{\tau}{T}\right)\frac{\sin\omega\tau}{\omega} - \left(\frac{-1}{T}\right)\left(-\frac{\cos\omega\tau}{\omega^2}\right)\right]_0^T$

$= 2\left[\left\{\frac{-1}{T}\cos\frac{\omega T}{\omega^2}\right\} - \left\{0 - \frac{1}{T}\cdot\frac{1}{\omega^2}\right\}\right]$

$= 2\left[\frac{1}{T\omega^2} - \frac{1}{T\omega^2}\cos\omega T\right]$

$\left(\begin{array}{l} 1-\cos 2\theta = 2\sin^2\theta \\ \therefore 1-\cos\theta = 2\sin^2\theta/2 \\ \therefore 1-\cos\omega T = 2\sin^2\frac{\omega T}{2} \end{array}\right.$

$S_{XX}(\omega) = \frac{2}{T\omega^2}(1-\cos\omega T) = \frac{2}{T\omega^2}\left(2\sin^2\frac{\omega T}{2}\right) = \frac{4}{T\omega^2}\sin^2\frac{\omega T}{2}.$

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14 b) (i) If the power spectral density of a Weak Sense Stationary process is given by $S(\omega) = \begin{cases} \frac{b}{a}(a-|\omega|) & , |\omega| \le a \\ 0 & , |\omega| > a \end{cases}$. Find the ACF of the process.

The auto correlation function $R_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega$

$= \frac{1}{2\pi}\left[\int_{-\infty}^{-a} S_{XX}(\omega)e^{j\omega\tau}d\omega + \int_{-a}^{a} S_{XX}(\omega)e^{j\omega\tau}d\omega + \int_{a}^{\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega\right.$.

$= \frac{1}{2\pi}\int_{-a}^{a} S_{XX}(\omega)e^{j\omega\tau}d\omega$ since $S_{XX}(\omega) = 0$ for $|\omega| > a$.

$= \frac{1}{2\pi}\int_{-a}^{a}\frac{b}{a}(a-|\omega|)e^{j\omega\tau}d\omega$

$= \frac{1}{2\pi}\int_{-a}^{a}\frac{b}{a}(a-|\omega|)\cos\omega\tau\, d\omega \quad \left(\because \int_{-a}^{a}\underbrace{\frac{b}{a}(a-|\omega|)\, j\sin\omega\tau\, d\omega}_{even\,\times\, odd\,=\,odd} = 0\right).$

$= \frac{2}{2\pi}\int_{0}^{a}\frac{b}{a}(a-\omega)\cos\omega\tau\, d\omega = \frac{b}{\pi a}\left[(a-\omega)\frac{\sin\omega\tau}{\tau} - (-1)\left(\frac{-\cos\omega\tau}{\tau^2}\right)\right]_0^a$

$= \frac{b}{\pi a}\left[\frac{1-\cos a\tau}{\tau^2}\right] = \frac{b}{\pi a\tau^2}(1-\cos a\tau) = \frac{b}{a\pi\tau^2}\, 2\sin^2\frac{a\tau}{2}$ //.

14 b) ii) The cross power density spectrum of the processes $\{X(t)\}$ and $\{Y(t)\}$ is given by
$$S_{XY}(\omega) = \begin{cases} a + jb\omega, & |\omega| < 1 \\ 0, & \text{otherwise} \end{cases}$$
where $\omega > 0$, a and b are real constants.

Find the cross correlation function $R_{xy}(\tau)$.

The cross-correlation function $R_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega) e^{j\omega\tau} d\omega$.

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{-1} S_{xy}(\omega) e^{j\omega\tau} d\omega + \int_{-1}^{1} S_{xy}(\omega) e^{j\omega\tau} d\omega + \int_{0}^{\infty} S_{xy}(\omega) e^{j\omega\tau} d\omega\right]$$

$$= \frac{1}{2\pi}\int_{-1}^{1} S_{xy}(\omega) e^{j\omega\tau} d\omega \quad [\because S_{xy}(\omega) = 0 \text{ in } |\omega| > 1]$$

$$= \frac{1}{2\pi}\left[a\int_{-1}^{1} e^{j\omega\tau} d\omega + jb\int_{-1}^{1} \omega e^{j\omega\tau} d\omega\right]$$

$$= \frac{a}{2\pi}\left[\frac{e^{j\omega\tau}}{j\tau}\right]_{-1}^{1} + \frac{jb}{2\pi}\left[\frac{\omega e^{j\omega\tau}}{j\tau} - \frac{e^{j\omega\tau}}{j^2\tau^2}\right]_{-1}^{1}$$

$$= \frac{1}{2\pi}\left[\frac{2a\sin\tau}{\tau} + \frac{2b\cos\tau}{\tau} - \frac{2b\sin\tau}{\tau^2}\right]$$

$$= \frac{1}{\pi\tau^2}\left[-b\sin\tau + b\tau\cos\tau + a\tau\sin\tau\right].$$

$$\underline{R_{xy}(\tau) = \frac{1}{\pi\tau^2}\left[(a\tau - b)\sin\tau + b\tau\cos\tau\right]}$$

15 a) i) Describe linear systems with random inputs and give an example.

The functional relationship between the input $x(t)$ and the output $y(t)$ is called a system i.e $y(t) = f(x(t))$, $-\alpha < t < \alpha$.
The system is said to be linear if $f(\alpha_1 x_1(t) + \alpha_2 x_2(t)) = \alpha_1 f(x_1(t)) + \alpha_2 f(x_2(t))$. A linear system is said to be time invariant if the input $x(t)$ is time shifted by an amount, then the corresponding output $y(t)$ should also be time shifted by the same amount. i.e if $y(t) = f(x(t))$ then $y(t+h) = f(x(t+h))$ then the system 'f' is called a time invariant system. The system $y(t) = f(x(t))$ is said to be a memoryless system if

$y(t)_{t\to t_1}$ depends only on $x(t)_{t\to t_1}$ not on the past or future values of $x(t)$. 8

i.e $y(t_1) = f(x(t_1))$.

The system is said to be casual if $y(t)$ at t_1 depends only on the past values of $x(t)$ up to $t = t_1$ i.e $y(t) = f(x(t))_{t \le t_1}$.

System in the form of Convolution:

If $h(t)$ is the system weighting function (unit impulse function) then the output $y(t)$ is given in terms of the convolution of $h(t)$ and input $x(t)$ as $y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du = \int_{-\infty}^{\infty} h(t-u)\, x(u)\, du$.

If $h(t)$ is the unit impulse response of the system, then the Fourier transform of $h(t)$ is the system transfer function $H(\omega)$,

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt \quad \text{and} \quad H^*(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{j\omega t}\, dt.$$

15(a) ii) Show that $S_{yy}(\omega) = S_{xx}(\omega)|H(\omega)|^2$ where $S_{xx}(\omega)$ and $S_{yy}(\omega)$ are the power spectral density functions of the input $x(t)$ and the output $y(t)$ and $H(\omega)$ is the system transfer function.

$$R_{xy}(\tau) = R_{xx}(\tau) * h(\tau)$$

Taking Fourier Transform, we have $F[R_{xy}(\tau)] = F[R_{xx}(\tau) * h(\tau)]$

$$F[R_{xy}(\tau)] = F[R_{xx}(\tau)]\, F[h(\tau)]$$

By convolution theorem on Fourier Transforms,

$$S_{xy}(\omega) = S_{xx}(\omega)\, F[h(\tau)] \quad \text{— ①} \qquad \text{(By definition of Spectrum).}$$

$$S_{xy}(\omega) = S_{xx}(\omega)\, H(\omega), \text{ where } H(\omega) = F(h(\tau))$$

$$R_{yy}(\tau) = R_{xy}(\tau) * h(-\tau)$$

$$F[R_{yy}(\tau)] = F[R_{xy}(\tau) * h(-\tau)]$$

$$= F[R_{xy}(\tau)]\, F[h(-\tau)]$$

$$S_{yy}(\omega) = S_{xy}(\omega)\, H^*(\omega) \quad \text{— ②}$$

From ① & ②, $S_{yy}(\omega) = S_{xx}(\omega)\, H(\omega)\, H^*(\omega)$

$$S_{yy}(\omega) = S_{xx}(\omega)|H(\omega)|^2.$$

15 b) (i) Show that, if $Y(t) = X(t+a) - X(t-a)$, $S_{YY}(W) = 4\sin^2 aw \, S_{XX}(W)$, where $X(t)$ is Weak Sense Stationary.

Given that $X(t)$ is a WSS random process. By definition,

$$R_{YY}(\tau) = E[Y(t)Y(t+\tau)] = E\{[X(t+a) - X(t-a)][X(t+\tau+a) - X(t+\tau-a)]\}$$

$$= E[X(t+a)X(t+\tau+a) - X(t+a)X(t+\tau-a) - X(t-a)X(t+\tau+a) + X(t-a)X(t+\tau-a)]$$

$$= R_{XX}(\tau) - R_{XX}(\tau-2a) - R_{XX}(\tau+2a) + R_{XX}(\tau)$$

$$= 2R_{XX}(\tau) - R_{XX}(\tau-2a) - R_{XX}(\tau+2a).$$

By definition of power spectral density function,

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau) e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} [2R_{XX}(\tau) - R_{XX}(\tau-2a) - R_{XX}(\tau+2a)] e^{-j\omega\tau} d\tau$$

$$= 2\int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau - \int_{-\infty}^{\infty} R_{XX}(\tau-2a) e^{-j\omega\tau} d\tau - \int_{-\infty}^{\infty} R_{XX}(\tau+2a) e^{-j\omega\tau} d\tau.$$

$$= 2S_{XX}(\omega) - \int_{-\infty}^{\infty} R_{XX}(\tau-2a) e^{-j\omega\tau} d\tau - \int_{-\infty}^{\infty} R_{XX}(\tau+2a) e^{-j\omega\tau} d\tau.$$

$$= 2S_{XX}(\omega) - e^{-j2\omega a} S_{XX}(\omega) - e^{j\omega 2a} S_{XX}(\omega).$$

$$= 2S_{XX}(\omega) - S_{XX}(\omega)(e^{j2\omega a} + e^{-j\omega 2a})$$

$$= 2S_{XX}(\omega) - S_{XX}(\omega)(2\cos 2\omega a)$$

$$= 2S_{XX}(\omega)(1 - \cos 2\omega a)$$

$$S_{YY}(\omega) = 4\sin^2 \omega a \, S_{XX}(\omega)$$

ii) State and prove the fundamental theorem on the power spectrum of output of a linear system.