April/ May 2011
Part-A

1. The cumulative distribution function of the random variable X is given by

$$F_X(x) = \begin{cases} \mathbf{0}, & x < 0 \\ x + \frac{1}{2}, & 0 \le x \le \frac{1}{2} \\ \mathbf{1}, & x > \frac{1}{2} \end{cases}$$

Compute $P\left[X > \frac{1}{4}\right]$.

Solution:

$$P\left[X > \frac{1}{4}\right] = P\left[\frac{1}{4} < X < \infty\right] = F[\infty] - F\left[\frac{1}{4}\right] = 1 - \left(\frac{1}{4} + \frac{1}{2}\right) = \frac{1}{4}$$

2. Let the random variable X denote the sum obtained in rolling a pair of dice. Determine the probability mass function of X.

Solution:

Probability mass function of X.

x:	2	3	4	5	6	7	8	9	10	11	12
$P(x)$:	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

3. Given the two regression lines $3X + 12Y = 19, 3Y + 9X = 46$, find the coefficient of correlation between X and Y.

Solution:

The regression lines of Y on X be $3X + 12Y = 19$

$$Y = \frac{1}{12}(19 - 3X) \dots (1)$$

The regression lines of Y on X be $3Y + 9X = 46$

$$X = \frac{1}{9}(46 - 3Y) \dots (2)$$

$$b_{yx} = Coefficient\ of\ x\ in\ (1) = -\frac{3}{12} = -\frac{1}{4}$$

$$b_{xy} = Coefficient\ of\ y\ in\ (2) = -\frac{3}{9} = -\frac{1}{3}$$

$$r^2 = b_{xy} b_{yx} = \frac{1}{12} \Rightarrow r = \pm 0.2887$$

$$r = -0.2887 \left(\because b_{xy} \text{ and } b_{yx} \text{ are negative}\right)$$

4. State central limit theorem.

Solution:

If $X_1, X_2, \ldots, X_n$ be a sequence of independent identically distributed random variables with $E(X_i) = \mu$ and $var\ (X_i) = \sigma^2 . i = 1,2, \ldots,$ and if $S_n = X_1 + X_2 + \cdots + X_n$ then S_n follows a normal distribution with mean $n\mu$ and variance $n\sigma^2$ as n tends to infinity.

5. Define (a) Continuous time random process. (b) Discrete state random process

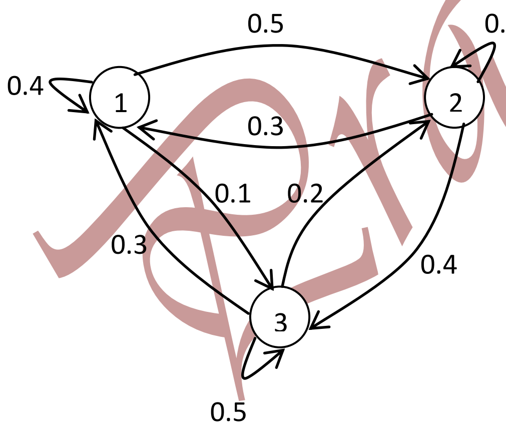
Solution:

Let $\{X(t, s), t \in T, s \in S\}$ be a random process.

(a) If both T and S are continuous, the random process is called a continuous time random process.

(b) If T is continuous and S is discrete, the random process is called a discrete random process.

6. Find the transition probability matrix of the process represented by the state transition diagram



Solution:

The transition probability matrix is given by

$$P = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

7. Arrival rate of telephone calls at a telephone booth is according to a Poisson distribution with an average time of 9 minutes between two consecutive arrivals. The length of a telephone call is assumed to be exponentially distributed with mean 3 minutes. Determine the probability that a person arriving at the booth will have to wait.

Solution:

$$\lambda = \frac{1}{9}\ per\ minutes$$

$$\mu = \frac{1}{3}\ per\ minutes$$

$$\rho = \frac{\lambda}{\mu} = \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{1}{3}$$

8. Trains arrive at the yard every 15 minutes and the service time is 33 minutes. If the line capacity of the yard is limited to 4 trains, find the probability that the yard is empty.

Solution:

$$N = 4, \lambda = \frac{1}{15}\ per\ minutes$$

$$\mu = \frac{1}{33}\ per\ minutes$$

$$\rho = \frac{\lambda}{\mu} = \frac{\frac{1}{15}}{\frac{1}{33}} = \frac{33}{15}$$

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-\left(\frac{33}{15}\right)}{1-\left(\frac{33}{15}\right)^5} = 0.0237$$

9. Given that the service time is Erlang with parameters $\boldsymbol{m}$ and $\boldsymbol{\mu}$. Show that the Pollaczek-Khintchine formula reduces to

$$\boldsymbol{L_s = m\rho + \frac{m(1+m)\rho^2}{2(1-m\rho)}}.$$

Solution:

Pollaczek-Khintchine formula for $M/G/1$ is

$$L_s = \lambda E(T) + \frac{\lambda^2\left(V(T) + E^2(T)\right)}{2\left(1-\lambda E(T)\right)} \qquad \dots(1)$$

$$Put\ \ E(T) = \frac{m}{\mu}\ and\ V(T) = \frac{m}{\mu^2}\ in\ (1)$$

$$L_s = \lambda\frac{m}{\mu} + \frac{\lambda^2\left(\frac{m}{\mu^2} + \left(\frac{m}{\mu}\right)^2\right)}{2\left(1 - \lambda\frac{m}{\mu}\right)}$$

$$L_s = m\rho + \frac{m(1+m)\rho^2}{2(1-m\rho)}. \qquad where\ \rho = \frac{\lambda}{\mu}$$

10. Give any two examples for series queueing situations.

Solution:

An assembly line in which units must pass through a series of work stations.

An admission process in a college where the students has to visit a series of officials or clerks.

Part-B

11. (a) (i) Find the moment-generating function of the binomial random variable with parameters m and p and hence find its means and variance.

Solution:

The probability mass function of binomial distribution is given by

$$P(X = x) = nC_x p^x q^{n-x}, x = 0,1,2, \dots$$

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} P(X = x)$$

$$= \sum_{x=0}^{\infty} e^{tx} nC_x p^x q^{n-x} = q^n \sum_{x=0}^{\infty} nC_x \left(\frac{pe^t}{q}\right)^x$$

$$= q^n \left[1 + n\left(\frac{pe^t}{q}\right) + \frac{n(n-1)}{2!}\left(\frac{pe^t}{q}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{pe^t}{q}\right)^3 + \cdots\right]$$

$$M_X(t) = q^n\left[1 + \frac{pe^t}{q}\right]^n = (q + pe^t)^n$$

$$M_X^{'}(t) = n(q+pe^t)^{n-1}pe^t$$

$$M_X^{''}(t) = n(n-1)(q+pe^t)^{n-1}(pe^t)^2 + n(q+pe^t)^{n-1}pe^t$$

$$E[x] = M_X^{'}(0) = n(q+pe^0)^{n-1}pe^0 = np(q+p) = np \qquad [\because p+q=1]$$

$$E[x^2] = M_X''(0) = n(n-1)(q+pe^0)^{n-1}(pe^0)^2 + n(q+pe^0)^{n-1}pe^t$$

$$E[x^2] = n(n-1)p^2 + np$$

$$Var(x) = E[x^2] - (E[x])^2 = n(n-1)p^2 + np - (np)^2 = np - np^2 = np(1-p) = npq$$

$$Mean = np, Variance = npq.$$

11. (a) (ii) Define Weibull distribution and write its mean and variance.
Solution:
A random variable X is said to have Weibull distribution with parameters α and β if its probability density function is given by

$$f(x) = \begin{cases} \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}, x > 0, \alpha > 0, \beta > 0 \\ \quad 0, \quad otherwise \end{cases}$$

$$Mean = \alpha^{-\frac{1}{\beta}}\Gamma\left(1+\frac{1}{\beta}\right)$$

$$Variance = \alpha^{-\frac{2}{\beta}}\left[\Gamma\left(1+\frac{2}{\beta}\right) - \left(\Gamma\left(1+\frac{1}{\beta}\right)\right)^2\right]$$

11.(b) (i) Derive mean and variance of a Geometric distribution. Also establish the forgetfulness property of the geometric distribution.
Solution:
The probability mass function of Geometric distribution is

$$P(X=x) = q^{x-1}p, x = 1,2,3, \ldots$$

Where $p + q = 1$

$$E[x] = \sum_{x=0}^{\infty} xP(X=x)$$

$$E[x] = \sum_{x=1}^{\infty} xq^{x-1}p = p[1 + 2q + 3q^2 + \cdots] = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$E[x^2] = \sum_{x=0}^{\infty} x^2P(X=x) = \sum_{x=0}^{\infty}[x(x+1) - x]P(X=x)$$

$$= \sum_{x=0}^{\infty} x(x+1)P(X=x) - \sum_{x=0}^{\infty} xP(X=x) = \sum_{x=1}^{\infty} x(x+1)q^{x-1}p - \sum_{x=1}^{\infty} xq^{x-1}p$$

$$E[x^2] = p[1.2 + 2.3\,q + 3.4\,q^2 + \cdots] - \frac{1}{p} = \frac{2p}{(1-q)^3} - \frac{1}{p} = \frac{2p}{p^3} - \frac{1}{p} = \frac{2}{p^2} - \frac{1}{p}$$

$$var(x) = E(x^2) - (E(x))^2 = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} = \frac{1-p}{p^2} = \frac{q}{p^2}$$

Memoryless property (forget fullness) of geometric distribution

$$P[X > m + n/X > m] = P[X > n]$$

11. (b) (ii) Suppose that telephone calls arriving at a particular switchboard follow a Poisson process with an average of 5 calls coming per minute. What is the probability that up to a minute will elapse unit 2 calls have come in to the switch boards?
Solution:
Poisson events following a Gamma distribution with $\lambda = 5$ and $k = 2$.

$$f(x) = \frac{\lambda^x x^{k-1} e^{-\lambda x}}{\Gamma(k)}, x \geq 0$$

$$f(x) = 25xe^{-5x}, x \geq 0$$

$$P(X \leq 1) = \int_0^1 25xe^{-5x} dx = 25\left[x\left(\frac{e^{-5x}}{-5}\right) - 1\left(\frac{e^{-5x}}{25}\right)\right]_0^1$$

$$= 25\left[\left(\frac{e^{-5}}{-5}\right) - \left(\frac{e^{-5}}{25}\right) + \frac{1}{25}\right] = 1 - 6e^{-5} = 0.9596.$$

12. (a) Given the joint density function

$$\boldsymbol{f(x,y)} = \begin{cases} x\dfrac{\mathbf{1+3y^2}}{\mathbf{4}}, \mathbf{0} < x < 2, 0 < y < 1 \\ \mathbf{0}, \quad \boldsymbol{elsewhere} \end{cases}$$

Find the marginal densities $\boldsymbol{g(x)}, \boldsymbol{h(y)}$ and the conditional density $\boldsymbol{f(x/y)}$ and evaluate

$$\boldsymbol{P}\left[\frac{\mathbf{1}}{\mathbf{4}} < x < \frac{\mathbf{1}}{\mathbf{2}} / \boldsymbol{y} = \frac{\mathbf{1}}{\mathbf{3}}\right]$$

Solution:

The marginal density function of X is given by

$$g(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1 \left(x\frac{1+3y^2}{4}\right)dy = \left(x\frac{y+y^3}{4}\right)_0^1 = \frac{1}{2}x, 0 < x < 2$$

The marginal density function of Y is given by

$$h(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^2 \left(x\frac{1+3y^2}{4}\right)dx = \left(\frac{1+3y^2}{4}\right)\left(\frac{x^2}{2}\right)_0^2 = \frac{1+3y^2}{2}, 0 < y < 1$$

$$f(x/y) = \frac{f(x,y)}{h(y)} = \frac{x\frac{1+3y^2}{4}}{\frac{1+3y^2}{2}} = \frac{x}{2}, 0 < x < 2$$

$$P\left[\frac{1}{4} < x < \frac{1}{2} / y = \frac{1}{3}\right] = \int_{\frac{1}{4}}^{\frac{1}{2}} f(x/y)dx = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{x}{2}dx = \frac{1}{2}\left(\frac{x^2}{2}\right)_{\frac{1}{4}}^{\frac{1}{2}} = \frac{1}{2}\left(\frac{1}{8} - \frac{1}{32}\right) = \frac{3}{64}$$

12. (b)(i) Determine whether the random variables X and Y are independent, given their joint probability density function as

$$\boldsymbol{f(x,y)} = \begin{cases} \boldsymbol{x^2} + \dfrac{\boldsymbol{xy}}{\mathbf{3}}, \mathbf{0} \leq \boldsymbol{x} \leq \mathbf{1}, \mathbf{0} \leq \boldsymbol{y} \leq \mathbf{2} \\ \mathbf{0}, \quad \boldsymbol{otherwise.} \end{cases}$$

Solution:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^2 \left(x^2 + \frac{xy}{3}\right)dy = \left(x^2y + \frac{xy^2}{6}\right)_0^2 = 2x^2 + \frac{2}{3}x, 0 \leq x \leq 1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^1 \left(x^2 + \frac{xy}{3}\right) dx = \left(\frac{x^3}{3} + \frac{x^2 y}{6}\right)_0^1 = \frac{1}{3} + \frac{1}{6}y, 0 \le y \le 2$$

$$f_X(x) f_Y(y) \ne f(x,y)$$

$\therefore X$ and Y are not independent.

12. (b) (ii) If X and Y are independent random variables having density functions

$$f(x) = \begin{cases} 2e^{-2x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

$$f(y) = \begin{cases} 3e^{-3y}, & y \ge 0 \\ 0, & y < 0 \end{cases}$$

respectively, find the density functions of $Z = X - Y$.

Solution:

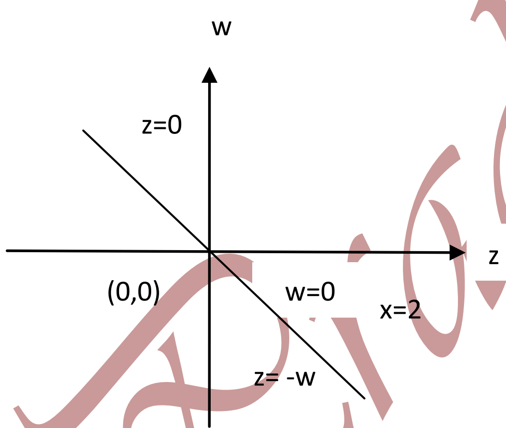
Let $Z = X - Y$ and $W = Y$

$$X = Z + Y = Z + W$$

The joint pdf of (U, V) is given by

$$f(z,w) = |J| f(x,y)$$

$$|J| = \begin{vmatrix} \frac{\partial X}{\partial Z} & \frac{\partial Y}{\partial Z} \\ \frac{\partial X}{\partial W} & \frac{\partial Y}{\partial W} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$



$$f(z,w) = |J| f(x,y) = f(x) f(y) \qquad [\because X \text{ and } Y \text{ are independent}]$$

$$f(z,w) = 2e^{-2x} 3e^{-3y} = 6e^{-2x-3y}, x \ge 0, y \ge 0.$$

$$f(z,w) = 6e^{-2(x-y)-5y} = 6e^{-2z-5w}, z \ge -w, w \ge 0$$

The density function of Z is given by

$$f_Z(z) = \int_{-z}^{\infty} 6e^{-2z-5w} dw, z < 0$$

$$= 6e^{-2z} \left[\frac{e^{-5w}}{-5}\right]_{-z}^{\infty} = \frac{6}{5} e^{3z}, z < 0$$

$$f_Z(z) = \int_0^{\infty} 6e^{-2z-5w} dw, z > 0$$

$$= 6e^{-2z}\left[\frac{e^{-5w}}{-5}\right]_0^\infty = \frac{6}{5}e^{-2z}, z > 0$$

$$f_Z(z) = \begin{cases} \frac{6}{5}e^{-2z}, & \boldsymbol{z \geq 0} \\ \frac{6}{5}e^{3z}, & \boldsymbol{z < 0} \end{cases}$$

13. (a) (i) Show that random process $X(t) = A\cos t + B\sin t, -\infty < t < \infty$ is a wide sense stationary process where A and B are independent random variables each of which has a value -2 with probability 1/3 and a value 1 with probability 2/3.

Solution:

$$E(A) = E(B) = \frac{1}{3}(-2) + \frac{2}{3}(1) = 0$$

$$E(A^2) = E(B^2) = \frac{1}{3}(-2)^2 + \frac{2}{3}(1)^2 = 2$$

Since A and B are independent random variables

$$E(AB) = E(A)E(B) = 0$$

$$E[X(t)] = E[A\cos t + B\sin t] = E(A)\cos t + E(B)\sin t = 0$$

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$$

$$= E\big[(A\cos t + B\sin t)\big(A\cos(t+\tau) + B\sin(t+\tau)\big)\big]$$

$$= E[A^2\cos t\cos(t+\tau) + B^2\sin t\sin(t+\tau) + AB\{\cos t\sin(t+\tau) + \sin t\sin(t+\tau)\}]$$

$$= E[A^2]\cos t\cos(t+\tau) + E[B^2]\sin t\sin(t+\tau) + E[AB]\{\cos t\sin(t+\tau) + \sin t\sin(t+\tau)\}$$

$$= 2\cos t\cos(t+\tau) + 2\sin t\sin(t+\tau) = 2\cos\tau$$

Mean is constant and auto correlation is a function of τ only.

$\therefore X(t)$ is a WSS.

13. (a) (ii) Derive probability distribution of Poisson process and hence find its correlation function.

Solution:

If $X(t)$represents the number of occurrences of a certain event in $(0, t)$ then the discrete random process $X(t)$ is called the Poisson process, provided the following postulates are satisfied.

(i) P[1 occurrence in $(t, t+\Delta t)] = \lambda\Delta t + O(\Delta t)$

(ii) P[0 occurrence in $(t, t+\Delta t)] = 1 - \lambda\Delta t + O(\Delta t)$

(iii) P[2 or more occurrences in $(t, t+\Delta t)] = O(\Delta t)$

(iv) $X(t)$ is independent of the number of occurrences of the event in any interval prior and after the interval $(0, t)$.

(v) The probability that the event occurs as specified number of times in $(t_0, t_0 + t)$ depends only on t, but not on t_0.

Probability law for the Poisson process $X(t)$:

Let λ be the number of occurrences of the event in unit time.

Let $P_n(t) = P[X(t) = n]$ =probability that there are n occurrences in $(0, t)$

$$P_n(t+\Delta t) = P\big((n-1)\ occurences\ in\ (0,t)\ and\ 1\ occurrences\ in\ (t, t+\Delta t)\big)$$

$$+P\big(n\ occurences\ in\ (0,t)\ and\ 0\ occurrences\ in\ (t, t+\Delta t)\big)$$

$$P_n(t+\Delta t) = P_{n-1}(t)\lambda\Delta t + P_n(t)(1-\lambda\Delta t)$$

$$P_n(t+\Delta t) = P_{n-1}(t)\lambda\Delta t + P_n(t) - P_n(t)\lambda\Delta t$$

$$P_n(t+\Delta t) - P_n(t) = P_{n-1}(t)\lambda\Delta t - P_n(t)\lambda\Delta t$$

$$\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = P_{n-1}(t)\lambda - P_n(t)\lambda$$

Taking limit as $\Delta t \to 0$, we get

$$\lim_{\Delta t \to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = P_{n-1}(t)\lambda - P_n(t)\lambda$$

$$\frac{d}{dt}P_n(t) = \lambda[P_{n-1}(t) - P_n(t)] \ldots (1)$$

Let the solution of (1) be

$$P_n(t) = \frac{(\lambda t)^n}{n!} f(t) \ldots (2)$$

Differentiating (2) with respect to t,

$$\frac{d}{dt}P_n(t) = \frac{\lambda^n}{n!}[nt^{n-1}f(t) + t^n f'(t)] \dots (3)$$

Using (2) and (3) in (1), we get

$$\frac{\lambda^n}{n!}[nt^{n-1}f(t) + t^n f'(t)] = \lambda\frac{(\lambda t)^{n-1}}{(n-1)!}f(t) - \lambda\frac{(\lambda t)^n}{n!}f(t)$$

$$\frac{\lambda^n}{n!}t^n f'(t) = -\lambda\frac{(\lambda t)^n}{n!}f(t) \Rightarrow f'(t) = -\lambda f(t) \Rightarrow f'(t) + \lambda f(t) = 0$$

$$f(t) = ke^{-\lambda t} \dots (4)$$

From (2) $f(0) =$ probability of number of occurrence in $(0,0) = P_0(0) = 1$

$$f(0) = ke^{-\lambda 0} \Rightarrow k = 1$$

$$f(t) = e^{-\lambda t} \dots (5)$$

Substituting (5) in (2), we get

$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, n -= 0,1,2,\dots$$

Auto correlation of the Poisson process:

$$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$$
$$= E[X(t_1)\{X(t_2) - X(t_1) + X(t_1)\}]$$
$$= E[X(t_1)\{X(t_2) - X(t_1)\}] + E[X^2(t_1)]$$
$$= E[X(t_1)]E[X(t_2) - X(t_1)] + E[X^2(t_1)]$$
$$= E[X(t_1)]\{E[X(t_2)] - E[X(t_1)]\} + E[X^2(t_1)]$$
$$= \lambda t_1(\lambda t_2 - \lambda t_1) + \lambda^2 t_1^2 + \lambda t_1$$
$$= \lambda^2 t_1 t_2 + \lambda t_1, t_2 > t_1$$
$$R_{XX}(t_1,t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1,t_2).$$

13. (b) (i) Find the limiting state probabilities associated with the following transition probability matrix.

$$\begin{bmatrix} \mathbf{0.4} & \mathbf{0.5} & \mathbf{0.1} \\ \mathbf{0.3} & \mathbf{0.3} & \mathbf{0.4} \\ \mathbf{0.3} & \mathbf{0.2} & \mathbf{0.5} \end{bmatrix}.$$

Solution:

Let $\pi = (\pi_1 \quad \pi_2 \quad \pi_3)$

$$P = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

$$\pi P = \pi \text{ and } \pi_1 + \pi_2 + \pi_3 = 1 \dots (1)$$

$$(\pi_1 \quad \pi_2 \quad \pi_3)\begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix} = (\pi_1 \quad \pi_2 \quad \pi_3)$$

$$0.4\,\pi_1 + 0.3\,\pi_2 + 0.3\pi_3 = \pi_1 \Rightarrow -0.6\,\pi_1 + 0.3\,\pi_2 + 0.3\pi_3 = 0 \ldots (2)$$

$$0.5\,\pi_1 + 0.3\,\pi_2 + 0.2\pi_3 = \pi_2 \Rightarrow 0.5\,\pi_1 - 0.7\,\pi_2 + 0.2\pi_3 = 0 \ldots (3)$$

$$0.1\,\pi_1 + 0.4\,\pi_2 + 0.5\pi_3 = \pi_3 \Rightarrow 0.1\,\pi_1 + 0.4\,\pi_2 - 0.5\pi_3 = 0 \ldots (4)$$

Solving (1), (2), (3) and (4) we get,

$$\pi_1 = 0.333, \pi_2 = 0.333, \pi_3 = 0.333$$

13. (b) (ii) Show that the difference of the two independent Poisson processes is not a Poisson process.

Solution:

$Let\ X(t) = X_1(t) - X_2(t)$

$Then\ E[X(t)] = E[X_1(t) - X_2(t)] = E[X_1(t)] - E[X_2(t)]$

$E[X(t)] = \lambda_1 t - \lambda_2 t = (\lambda_1 - \lambda_2)t$

$E[X^2(t)] = E\left[\left(X_1(t) - X_2(t)\right)^2\right]$

$= E[X_1^2(t) + X_2^2(t) - 2X_1(t)X_2(t)]$

$= E[X_1^2(t)] + E[X_2^2(t)] - 2E[X_1(t)X_2(t)]$

$= E[X_1^2(t)] + E[X_2^2(t)] - 2E[X_1(t)]E[X_2(t)] \qquad (Since\ X_1(t)\ \&\ X_2(t)\ are\ independent)$

$= (\lambda_1 t)^2 + \lambda_1 t + (\lambda_2 t)^2 + \lambda_2 t - 2\lambda_1 t \lambda_2 t = (\lambda_1 - \lambda_2)^2 t^2 + (\lambda_1 + \lambda_2)t$

$E[X^2(t)] = (\lambda_1 - \lambda_2)^2 t^2 + (\lambda_1 + \lambda_2)t \neq (\lambda_1 - \lambda_2)^2 t^2 + (\lambda_1 - \lambda_2)t$

Hence, the difference of the two independent Poisson processes is not a Poisson process.

14. (a) (i) Customers arrive at a one window drive-in bank according to Poisson distribution with
mean 10 per hour. Service time per customer is exponential with mean 5 minutes. The space is front of window, including that for the serviced car can accommodate a maximum of three cars. Others cars can wait outside this space.
(1) What is the probability that an arriving customer can drive directly to the space in front of the window?
(2) What is the probability that an arriving customer will have to wait outside the indicated space?
(3) How long is an arriving customer expected to wait before being served?

Solution:

This problem is of the model $(M/M/1):(\infty/FIFO)$

$\lambda = 10\ customers/hr$

$\mu = 5\ min/customer = \frac{1}{5}\ cutomer/min = \frac{60}{5} = 12\ customers/hr$

$\rho = \frac{\lambda}{\mu} = \frac{10}{12} = \frac{5}{6}$

1) The probability that an arriving customer can drive directly to the space in front of the window is

$P(X < 3) = 1 - P(X \geq 3) = 1 - \rho^N = 1 - \left(\frac{5}{6}\right)^3 = 0.4213$

2) The probability that an arriving customer will have to wait outside the indicated space

$P(N \geq 3) = \rho^N = \left(\frac{5}{6}\right)^3 = 0.5787$

3) Arriving customer expected to wait before being served

$$W_q = \frac{L_q}{\lambda} = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{10}{12(12-10)} = \frac{5}{12}\ hours$$

14.(a)(ii) Show that for the $(M/M/1):(FCFS/\infty/\infty)$, the distribution of waiting time in the system is

$$\boldsymbol{W(t) = (\mu - \lambda)e^{-(\mu-\lambda)t}, t > 0}$$

Solution:

$$W(t) = \sum_{n=0}^{\infty} W(t/n+1)\, P_n$$

$$W(t) = \sum_{n=0}^{\infty} \frac{\mu(\mu t)^n e^{-\mu t}}{n!} (1-\rho)\rho^n \ where\ \rho = \frac{\lambda}{\mu}$$

$$= \mu e^{-\mu t}(1-\rho) \sum_{n=0}^{\infty} \frac{(\mu\rho t)^n}{n!} = \mu e^{-\mu t}\left(1 - \frac{\lambda}{\mu}\right) \sum_{n=0}^{\infty} \frac{\left(\mu \frac{\lambda}{\mu} t\right)^n}{n!}$$

$$= \mu e^{-\mu t}\left(1 - \frac{\lambda}{\mu}\right) \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} = \mu e^{-\mu t}\left(1 - \frac{\lambda}{\mu}\right) e^{\lambda t} \qquad \left[Since \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x\right]$$

$$\therefore W(t) = (\mu - \lambda)e^{-(\mu-\lambda)t}$$

14. (b) Find the steady state solution for the multiserver $(M/M/C)$ model and hence find L_q, W_q, L_s and W_s by using Little's formula.

Solution:

Let $\lambda_n = \lambda$ for all n.

For the Poisson queue system

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n}} \quad \dots (1)$$

$$P_n = \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n} P_0, n \geq 1 \quad \dots (2)$$

If there is a single server, $\mu_n = \mu$ for all n. But there are C servers working independently of each other. If there be less than C customer. i.e., if $n < C$, only n of the C servers will be busy and the others idle and hence the mean service rate will be null.

If $n \geq C$ all the C serves will be busy and hence the mean source rate $= C\mu$

$$\therefore \mu_n = \begin{cases} n\mu, & 0 \leq n < C \\ C\mu, & n \geq C \end{cases} \quad \dots (3)$$

Using (3) in (1) and (2) we get,

$$P_n = \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n P_0, \quad 0 \leq n < C \dots (4)$$

$$P_n = \frac{\lambda^n P_0}{1.2\mu.3\mu \dots (C-1)\mu.(C\mu)(C\mu)(C\mu) \dots (n-C+1)times}, n \geq C$$

$$P_n = \frac{\lambda^n P_0}{(c-1)!\,\mu^{C-1} (C\mu)^{(n-C+1)}}, n \geq c$$

$$P_n = \frac{1}{C!\,(C)^{(n-C)}} \left(\frac{\lambda}{\mu}\right)^n P_0, n \geq c \dots (6)$$

Now P_0 is given by

$$\sum_{n=0}^{\infty} P_n = 1$$

$$\left[\sum_{n=0}^{C-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=C}^{\infty} \frac{1}{C!\,(C)^{(n-C)}}\left(\frac{\lambda}{\mu}\right)^n\right] P_0 = 1$$

$$\left[\sum_{n=0}^{C-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=C}^{\infty} \frac{(C)^C}{C!}\left(\frac{\lambda}{\mu C}\right)^n\right] P_0 = 1$$

$$\left[\sum_{n=0}^{C-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{(C)^C}{C!}\left(\frac{\lambda}{\mu C}\right)^C \left(\frac{1}{1 - \frac{\lambda}{\mu C}}\right)\right] P_0 = 1$$

$$\left[\sum_{n=0}^{C-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n+\frac{1}{C!\left(1-\frac{\lambda}{\mu C}\right)}\left(\frac{\lambda}{\mu}\right)^C\right]P_0=1$$

$$P_0=\left[\sum_{n=0}^{C-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n+\frac{1}{C!\left(1-\frac{\lambda}{\mu C}\right)}\left(\frac{\lambda}{\mu}\right)^C\right]^{-1}\quad\ldots(6)$$

$$L_q=E[n-C]=\sum_{n=C}^{\infty}(n-C)P_n$$

$$=\sum_{n=C}^{\infty}(n-C)\frac{1}{C!\,(C)^{(n-C)}}\left(\frac{\lambda}{\mu}\right)^n P_0$$

$$=\frac{(C)^C}{C!}P_0\sum_{n=C}^{\infty}(n-C)\left(\frac{\lambda}{\mu C}\right)^n$$

$$=\frac{(C)^C}{C!}P_0\left[\left(\frac{\lambda}{\mu C}\right)^{C+1}+2\left(\frac{\lambda}{\mu C}\right)^{C+2}+3\left(\frac{\lambda}{\mu C}\right)^{C+3}+\cdots\right]$$

$$=\frac{(C)^C}{C!}\left(\frac{\lambda}{\mu C}\right)^{C+1}P_0\left[1+2\left(\frac{\lambda}{\mu C}\right)^1+3\left(\frac{\lambda}{\mu C}\right)^2+\cdots\right]$$

$$=\frac{(C)^C}{C!}\left(\frac{\lambda}{\mu C}\right)^{C+1}\frac{1}{\left(1-\frac{\lambda}{\mu C}\right)^2}P_0$$

$$L_s=L_q+\frac{\lambda}{\mu}=\frac{(C)^C}{C!}\left(\frac{\lambda}{\mu C}\right)^{C+1}\frac{1}{\left(1-\frac{\lambda}{\mu C}\right)^2}P_0+\frac{\lambda}{\mu}$$

$$W_s=\frac{L_s}{\lambda}=\frac{(C)^C}{\lambda\,C!}\left(\frac{\lambda}{\mu C}\right)^{C+1}\frac{1}{\left(1-\frac{\lambda}{\mu C}\right)^2}P_0+\frac{1}{\mu}$$

$$W_q=\frac{L_q}{\lambda}=\frac{(C)^C}{\lambda\,C!}\left(\frac{\lambda}{\mu C}\right)^{C+1}\frac{1}{\left(1-\frac{\lambda}{\mu C}\right)^2}P_0$$

15. (a) Derive the expected steady state system size for the single server queues with Poisson input and General service or Derive Pollaczek-Khintchine formula.

Solution:

Let n and n_1 be the number of customer in the system at times t and $t+T$, when two consecutive customers have just left the system after getting service.

Let $f(t), E(T)$ and $var(T)$ be the probability density function, mean and variance of T. Let k be the number of customers arriving in the system during the service time T.

Hence

$$n_1 = \begin{cases} k, & if\ n=0 \\ n-1+k, & if\ n>0 \end{cases}$$

Where $k = 0,1,2,3, \dots,$ is the number of arrivals during the service time. If

$$\delta = \begin{cases} 1 & if\ n=0 \\ 0 & if\ n>1 \end{cases}$$

Then $n_1 = n - 1 + \delta + k$ $\quad\quad\quad\quad\quad \dots(1)$

$$E(n_1) = E(n-1+\delta+k) \Rightarrow E(n_1) = E(n) - 1 + E(\delta) + E(k) \quad \dots(2)$$

When the system has reached the steady state, the probability of the number of customers in the system will be constant. Hence

$$E(n_1) = E(n)\ and\ E(n_1^2) = E(n^2) \quad \dots(3)$$

Substituting (3) in (2), we get

$$-1 + E(\delta) + E(k) = 0 \Rightarrow E(\delta) = 1 - E(k) \quad \dots(4)$$

$$n_1^2 = (n + k - 1 + \delta)^2$$

$$n_1^2 = n^2 + (k-1)^2 + \delta^2 + 2n(k-1) + 2n\delta + 2\delta(k-1)$$

$$n_1^2 = n^2 + k^2 - 2k + 1 + \delta^2 + 2n(k-1) + 2n\delta + 2\delta k - 2\delta \dots(5)$$

Since $\delta = \delta^2$ and n $\delta = 0$,we get

$$n_1^2 = n^2 + k^2 - 2k + 1 + \delta + 2n(k-1) + 2\delta k - 2\delta$$

$$2n(1-k) = n^2 - n_1^2 + k^2 - 2k + 1 + 2\delta k - \delta$$

$$2n(1-k) = n^2 - n_1^2 + k^2 - 2k + 1 + \delta(2k-1)$$

$$E\big(2n(1-k)\big) = E\big(n^2 - n_1^2 + k^2 - 2k + 1 + \delta(2k-1)\big)$$

$$2E(n)E(1-k) = E(n^2) - E(n_1^2) + E(k^2) - 2E(k) + 1 + E(\delta)E(2k-1)$$

$$2E(n)\big(1-E(k)\big) = E(k^2) - 2E(k) + 1 + \big(1-E(k)\big)E(2k-1) \quad \text{(Using (3)\& (4))}$$

$$2E(n)\big(1-E(k)\big) = E(k^2) - 2E(k) + 1 + \big(1-E(k)\big)\big(2E(k)-1\big)$$

$$E(n)=\frac{\left(E(k^2)-2E(k)+1+\left(1-E(k)\right)(2E(k)-1)\right)}{2\left(1-E(k)\right)}$$

$$E(n)=\frac{E(k^2)+\left(-1+1-E(k)\right)(2E(k)-1)}{2\left(1-E(k)\right)}$$

$$E(n)=\frac{E(k^2)+\left(-E(k)\right)(2E(k)-1)}{2\left(1-E(k)\right)}=\frac{E(k^2)+E(k)-2E^2(k)}{2\left(1-E(k)\right)}$$

$$E(n)=\frac{E(k^2)-E^2(k)+E(k)-E^2(k)}{2\left(1-E(k)\right)}=\frac{Var\ (k)+E(k)-E^2(k)}{2\left(1-E(k)\right)}\ldots(6)$$

Since the number k of arrivals follows Poisson process with parameter λ.

$$E(k/T)=\lambda T$$

$$E(k^2/T)=\lambda^2T^2+\lambda T$$

$$E(k)=\int_0^\infty E(k/T)\,f(t)dt=\lambda\int_0^\infty T\,f(t)dt=\lambda E(T)\ldots(7)$$

$$E(k^2)=\int_0^\infty E(k^2/T)\,f(t)dt=\int_0^\infty(\lambda^2T^2+\lambda T)\,f(t)dt=\lambda^2\int_0^\infty T^2\,f(t)dt+\lambda\int_0^\infty T\,f(t)dt$$

$$E(k^2)=\lambda^2E(T^2)+\lambda E(T)\ldots(8)$$

$$Var(k)=E(k^2)-E^2(k)=\lambda^2E(T^2)+\lambda E(T)-\lambda^2E^2(T)=\lambda^2\left(E(T^2)-E^2(T)\right)+\lambda E(T)$$

$$Var(k)=\lambda^2Var(T)+\lambda E(T)\ldots(9)$$

Substituting (7),(8) and (9) in (6), we get

$$E(n)=\frac{\lambda^2Var(T)+\lambda E(T)+\lambda E(T)-\lambda^2E^2(T)}{2\left(1-\lambda E(T)\right)}$$

$$E(n)=\frac{\lambda^2Var(T)+\lambda^2E^2(T)+2\lambda E(T)-2\lambda^2E^2(T)}{2\left(1-\lambda E(T)\right)}$$

$$E(n)=\frac{\lambda^2\left(Var(T)+E^2(T)\right)+2\lambda E(T)\left(1-\lambda E(T)\right)}{2\left(1-\lambda E(T)\right)}$$

$$E(n)=\lambda E(T)+\frac{\lambda^2\left(Var(T)+E^2(T)\right)}{2\left(1-\lambda E(T)\right)}$$

15. (b) Write Short notes on :

(i) Series Queues

(ii) Open and Closed Queue networks

Series Queues:

A special type of open queueing network called series queue.

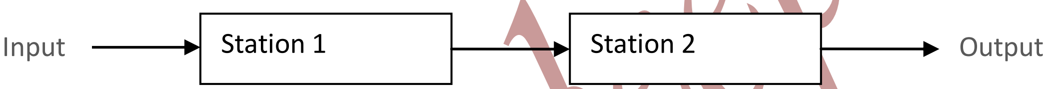
In open network there are a series of service facilities which each customer should visit in the given order before leaving the system. The nodes form a series system flow always in a single direction from node to node. Customers enter from outside only at node 1 and depart only from node k.

Example: Registration process in university, in clinic physical examination procedure.

There are two types of series queue.

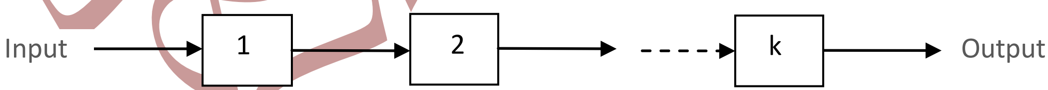
Series queues with blocking:

A queueing system with two stations in series. One server if each station and no queue allowed to form at either station. A customer entering for service has to go through station 1 and then station 2. No queues are allowed in front of station 1 and station 2.



Series Queues with infinite capacity:

A System with k stations in series. Arrivals at station 1 are generated from an infinite population, according to a Poisson distribution with mean arrival rate λ. Serviced units will move successively from one station to the next until they depart from last station. Service time distribution of each station I is exponential with mean rate $\mu_i, i = 1,2,\ldots,k$. There is no queue limit at any station.



Open and Closed Queue networks

In a open queueing network, customers may arrive from outside the system at any node and may depart from the system from any node.

In a closed queueing network, customers are not allowed to enter from outside or to leave the system.

In open Jackson network, customers arrive at each station both from outside the system and from other stations. The customers may visit the various stations in any order and may skip some stations. Each station has a infinite queue capacity and may have multiple servers. A customer leaving station i goes to station j with probability P_{ij}.