MA2211 - TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

S.KANAKALAKSHMI

APRIL / MAY 2011.

1. Write expression for the Fourier series co-efficient b_n for the function $f(x)$ defined in $(-2,2)$.

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx \qquad \text{Here } (-l,l) = (-2,2)$$

$$\therefore \boxed{l=2}$$

$$\therefore b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx.$$

2. Without finding the values of a_0, a_n and b_n, the Fourier coefficients of Fourier series for the function $f(x)=x^2$ in the interval $(0,\pi)$ find the value of $\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)$.

R.M.S value of $f(x)$ is,

$$\bar{y}^2 = \frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2), \quad \bar{y}^2 = \frac{1}{b-a}\int_a^b (f(x))^2\,dx.$$

$$\therefore \frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2) = \frac{2}{\pi}\int_0^{\pi}(x^2)^2\,dx.$$

$$= \frac{2}{\pi}\left[\frac{x^5}{5}\right]_0^{\pi}$$

$$= \frac{2}{\cancel{\pi}}\cdot\frac{\pi^{\cancel{5}\,4}}{5}$$

i.e $$\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2) = \frac{2\pi^4}{5}$$

3. State and prove change of scale property of Fourier transform.

$$F[f(ax)] = \frac{1}{a}F(s/a),\ a>0.$$

Proof: $$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)\,e^{isx}\,dx$$

Put $ax = y$ when $x = -\infty$, $y = -\infty$

$a\,dx = dy$ and $x = \infty$, $y = \infty$.

$\Rightarrow dx = \frac{dy}{a}$

$$\therefore F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, e^{isy/a} \cdot \frac{dy}{a}$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, e^{i(s/a)y}\, dy$$

$$= \frac{1}{a} F(s/a)$$

$$\therefore F[f(ax)] = \frac{1}{a} F(s/a),\ a > 0.$$

4. If $F_c(s)$ is the Fourier cosine transform of $f(x)$, prove that the Fourier cosine transform of $f(ax)$ is $\frac{1}{a} F_c(s/a)$.

Fourier cosine transform

$$F_c[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(ax) \cos sx\, dx \quad = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(y) \cos\left(\frac{sy}{a}\right) \cdot \frac{dy}{a}$$

Put $ax = y$ | when $x = 0$, $y = 0$

$a\,dx = dy$ | & $x = \infty$, $y = \infty$.

$dx = \frac{dy}{a}$

$$F_c[f(ax)] = \frac{1}{a} \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(y) \cos(s/a)y \cdot dy$$

$$\therefore F_c[f(ax)] = \frac{1}{a} F_c(s/a)$$

5. Form the partial differential equations by eliminating the arbitrary constant a and b from $z = (x^2+a^2)(y^2+b^2)$.

Given $z = (x^2+a^2)(y^2+b^2)$ — ①

Differentiating ① w.r.to 'x' we get

$$\frac{\partial z}{\partial x} = (y^2+b^2) \cdot 2x \quad \Rightarrow y^2+b^2 = \frac{p}{2x} \quad \text{— ②}$$

Differentiating ① partially w.r.to 'y'

$$\frac{\partial z}{\partial y} = (x^2+a^2) \cdot 2y \quad \Rightarrow x^2+a^2 = \frac{q}{2y} \quad \text{— ③}$$

using ② and ③ in ① we get,

$$z = \frac{q}{2y}\frac{p}{2x}$$

$$4xyz = pq$$

6. Solve the equation $(D-D')^3 z = 0$.

Auxiliary equation is $(m-1)^3 = 0 \Rightarrow m-1=0$ (three times)

i.e $m=1$ (three times)

All the three roots are same.

$\therefore z = f_1(y+x) + x f_2(y+x) + x^2 f_3(y+x)$.

7. A rod 40cm long with insulated sides has its ends A and B kept at 20°c and 60°c respectively. Find the steady state temperature at a location 15 cm from A.

Given l = length of the rod = 40cm.

The steady state temperature at any point is $u(x) = ax+b$ —①.

When steady state conditions exists the boundary conditions are

i) $u(0) = 20°c$

ii) $u(l) = 60°c$, $l = 40$

Using (i) in ①, $\underbrace{u(0)}_{20°c} = 0+b \Rightarrow \boxed{b = 20°}$

Using (ii) in ①, $u(40) = 40a+b \Rightarrow 60°c = 40a + 20°$

$40a = 60° - 20°c \Rightarrow 40a = 40 \Rightarrow \boxed{a=1}$

$\therefore u(x) = x + 20$

8. Write down the three possible solutions of Laplace equation in two dimensions.

(i) $u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$

(ii) $u(x,y) = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py})$

(iii) $u(x,y) = (c_9 x + c_{10})(c_{11} y + c_{12})$.

9. Find the Z-transform of a^n.

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$Z[a^n] = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \frac{a^n}{z^n} = \sum_{n=0}^{\infty} (a/z)^n = 1 + a/z + (a/z)^2 + \cdots$$

$$= (1 - a/z)^{-1} = \frac{1}{1-a/z} = \frac{1}{\frac{z-a}{z}} = \frac{z}{z-a}$$

$$\therefore Z[a^n] = \frac{z}{z-a}$$

10. What advantage is gained when Z-transform is used to solve difference equation?

Z-transform method for solving linear difference equation is analogous to the Laplace transform method for solving linear constant differential equations. Such difference equations can be solved directly in MAPLE using the rsolve command.

—x—x—

Part B

11. a) i) Expand $f(x) = x(2\pi - x)$ as Fourier series in $(0, 2\pi)$ and hence deduce that the sum of $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

The fourier series of $f(x)$ in $(0, 2\pi)$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$.

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2)dx = \frac{1}{\pi}\left[\frac{2\pi x^2}{2} - \frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{\pi}\left[4\pi^3 - \frac{8\pi^3}{3}\right]$

$$\boxed{a_0 = \frac{4\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2) \cos nx\,dx$$

$$= \frac{1}{\pi}\left[(2\pi x - x^2)\frac{\sin nx}{n} + (2\pi - 2x)\frac{\cos nx}{n^2} + 2\frac{\sin nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left\{0 + \left(\frac{-2\pi}{n^2}\right) + 0\right\} - \left\{0 + \frac{2\pi}{n^2}\right\}\right]$$

$$= \frac{-4\cancel{\pi}}{\cancel{\pi} n^2}$$

$$\boxed{a_n = -4/n^2}$$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{\pi}\int_0^{2\pi} \underbrace{(2\pi x - x^2)}_{u}\underbrace{\sin nx}_{v}\,dx$

$= \frac{1}{\pi}\left[-(2\pi x - x^2)\frac{\cos nx}{n} + (2\pi - 2x)\frac{\sin nx}{n^2} + (-2)\frac{\cos nx}{n^3}\right]_0^{2\pi}$

$= \frac{1}{\pi}\left[\left\{0 + 0 + \frac{-2}{n^3}\right\} - \left\{0 + 0 + \frac{-2}{n^3}\right\}\right] = 0.$

$\therefore x(2\pi - x) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\frac{-4}{n^2}\cos nx.$ is the required series.

put $x=0$ in the series we get,

$0 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\frac{-4}{n^2}$

$\cancel{-}\frac{\cancel{2}\pi^2}{3} = \cancel{-}\cancel{4}_{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$

$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$

(ii) Obtain the Fourier series for the function $f(x)$ given by

$f(x) = \begin{cases} 1-x, & -\pi < x < 0 \\ 1+x, & 0 < x < \pi \end{cases}$. Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \pi^2/8$.

Given $f(x) = \begin{cases} 1-x, & -\pi < x < 0 \\ 1+x, & 0 < x < \pi \end{cases}$

Here the interval is symmetric.

$\phi_1(x) = 1-x \quad , \phi_2(x) = 1+x$

$\phi_1(-x) = 1+x = \phi_2(x).$ $\quad \therefore f(x)$ is an even fn. $\boxed{\therefore b_n = 0}$

and $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$ and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$

i.e $a_0 = \frac{2}{\pi}\int_0^{\pi}(1+x)\,dx = \frac{2}{\pi}\left[x + \frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\left[\pi + \frac{\pi^2}{2}\right] = \frac{2}{\cancel{\pi}}\cdot\cancel{\pi}(1+\pi/2)$

$\boxed{a_0 = 2(1+\pi/2)}$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(1+x)\cos nx\,dx = \frac{2}{\pi}\left[(1+x)\frac{\sin nx}{n} - 1\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{0+\frac{(-1)^n}{n^2}\right\} - \left\{0+\frac{1}{n^2}\right\}\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$a_n = \begin{cases} 0, & n = \text{even} \\ \frac{-4}{n^2\pi}, & n = \text{odd}. \end{cases}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$\text{ie } f(x) = (1+\pi/2) + \sum_{n=1,3}^{\infty}\frac{-4}{n^2\pi}\cos nx$$

11.b)(i) Obtain the sine series for $f(x) = \begin{cases} x & \text{in } 0 \le x \le l/2. \\ l-x & \text{in } l/2 \le x \le l. \end{cases}$

The sine series of $f(x)$ is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}.$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$

$$= \frac{2}{l}\left[\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right].$$

$$= \frac{2}{l}\left[\left(-x\cos\frac{n\pi x}{l}\times\frac{l}{n\pi} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)_0^{l/2} + \left(-(l-x)\cos\frac{n\pi x}{l}\times\frac{l}{n\pi} - \frac{l^2}{n^2\pi^2}(-1)\left(-\frac{\sin n\pi x}{l}\right)\right)_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\left(\left\{-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\} - \{0+0\}\right) + \left(\{0+0\} - \left\{-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}\right)\right]$$

$$= \frac{2}{\cancel{l}}\left[\frac{2l^{\cancel{2}}}{n^2\pi^2}\sin\frac{n\pi}{2}\right] = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}. \quad \therefore f(x) = \sum_{n=1}^{\infty}\frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}.$$

11 b) (ii) Find the Fourier series upto second harmonic for $y = f(x)$ from the following values. (4)

x: 0 $\pi/3$ $2\pi/3$ π $4\pi/3$ $5\pi/3$ 2π

y: 1.0 1.4 1.9 1.7 1.5 1.2 1.0.

Solution:

The Fourier Series is given by $y = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x)$

$a_0 = 2\left(\frac{\Sigma y}{n}\right)$; $a_1 = 2\left(\frac{\Sigma y \cos x}{n}\right)$; $a_2 = 2\left(\frac{\Sigma y \cos 2x}{n}\right)$

$b_1 = 2\left(\frac{\Sigma y \sin x}{n}\right)$; $b_2 = 2\left(\frac{\Sigma y \sin 2x}{n}\right)$

Here $n = 6$.

x	y	$\cos x$	$\sin x$	$\cos 2x$	$\sin 2x$	$y\cos x$	$y\sin x$	$y\cos 2x$	$y\sin 2x$
0	1.0	1	0	1	0	1	0	1	0
$\pi/3$	1.4	0.5	0.866	0.5	0.866	0.7	1.212	-0.7	1.212
$2\pi/3$	1.9	-0.5	0.866	-0.5	-0.866	-0.95	1.65	-0.95	-1.645
π	1.7	-1	0	-1	0	-1.7	0	1.7	0
$4\pi/3$	1.5	-0.5	-0.866	-0.5	0.866	-0.75	-1.299	-0.75	1.299
$5\pi/3$	1.2	0.5	-0.866	-0.5	-0.866	0.6	-1.039	-0.6	-1.039
Σ	8.7					-1.1	0.5196	-0.3	-0.1732.

Here $n = 6$.

$a_0 = \frac{2}{6} \times 8.7 = 2.9$; $a_1 = 2\left(\frac{\Sigma y \cos x}{6}\right) = -0.37$

$a_2 = 2\left(\frac{\Sigma y \cos 2x}{n}\right) = \frac{2}{6} \times -0.3 = -0.1$

$b_1 = 2\left(\frac{\Sigma y \sin x}{n}\right) = \frac{2}{6} \times 0.5196 = 0.17$

$b_2 = 2\left(\frac{\Sigma y \sin 2x}{n}\right) = \frac{2}{6} \times -0.1732 = -0.06.$

$\therefore y = 1.45 + (-0.37\cos x + 0.17\sin x) - (0.1\cos 2x + 0.06\sin 2x)$

12 a) i) Find the Fourier transform of $f(x) = \begin{cases} 1-x^2 & \text{if } |x|<1 \\ 0 & \text{if } |x|>1 \end{cases}$. Hence evaluate $\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx$.

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-1} f(x)e^{isx}dx + \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} f(x)e^{isx}dx + \frac{1}{\sqrt{2\pi}}\int_{1}^{\infty} f(x)e^{isx}dx.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-1} 0\,e^{isx}dx + \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)e^{isx}dx + \frac{1}{\sqrt{2\pi}}\int_{1}^{\infty} 0.e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[(1-x^2)\frac{e^{isx}}{is} - (-2x)\frac{e^{isx}}{i^2s^2} - 2\frac{e^{isx}}{i^3s^3}\right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-2}{s^2}e^{is} + \frac{2}{is^3}e^{is} + \frac{-2}{s^2}e^{-is} - \frac{2}{i}\frac{e^{-is}}{s^3}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-2}{s^2}(e^{is}+e^{-is}) + \frac{2}{is^3}(e^{is}-e^{-is})\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-4}{s^2}\cos s + \frac{4}{s^3}\sin s\right]$$

$$\therefore F(s) = \frac{-1}{\sqrt{2\pi}}\frac{4}{s^3}\left(\frac{s\cos s - \sin s}{s^3}\right)$$

By using inverse Fourier transform we get,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{-4}{\sqrt{2\pi}s^3}\left(\frac{s\cos s - \sin s}{s^3}\right).e^{-isx}ds.$$

$$= \frac{-1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{s^3}\,(s\cos s - \sin s)\,(\cos x - i\sin sx)\,ds.$$

$$= \frac{-2}{\pi}\int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}\cos sx\,dx \;-0$$

$\left(\because \frac{4}{s^3}(s\cos s-\sin s)\sin sx \to \text{odd function}\right.$

$\int_{-\infty}^{\infty}\frac{4}{s^3}(s\cos s-\sin s)\sin sx\,ds=0$

$\frac{s\cos s-\sin s}{s^3}\cos sx \to$ even function

$\int_{-\infty}^{\infty}4\left(\frac{s\cos s-\sin s}{s^3}\right)\cos sx\,dx = 2\int_0^{\infty}4\left(\frac{s\cos s-\sin s}{s^3}\right)\cos sx\,dx$

$$\therefore f(x) = \frac{-4}{\pi}\int_0^{\infty}\frac{s\cos s - \sin s}{s^3}\cos s x\,ds.$$ ie integrand even.

$$\int_0^{\infty}\frac{s\cos s - \sin s}{s^3}\cos sx\,ds = \frac{-\pi}{4}f(x).$$

Put $x = 1/2$ we get

$$\int_0^{\infty}\frac{s\cos s - \sin s}{s^3}\cos s/2\,ds = \frac{-\pi}{4}f(1/2)$$

$$= \frac{-\pi}{4}(1-1/4)$$

$$= \frac{-3\pi}{16}$$

Replacing s by x we get,

$$\int_0^{\infty}\left(\frac{x\cos x - \sin x}{x^3}\right)\cos x/2\,dx = \frac{-3\pi}{16}$$

(ii) Find the Fourier transform of $f(x)$ given by $f(x) = \begin{cases}1 & \text{for } |x|<a\\ 0 & \text{for } |x|>a>0\end{cases}$ and using Parseval's identity prove that $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \pi/2$.

Given $f(x) = 1$, $-a<x<a$

$f(x)=0$, $-\infty<x<-a$ and $a<x<\infty$.

$$\therefore F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}e^{isx}\,dx = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isx}}{is}\right]_{-a}^{a} = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isa}}{is} - \frac{e^{-isa}}{is}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isa}-e^{-isa}}{is}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2i\sin as}{is}\right]$$

$$F(s) = \sqrt{\frac{2}{\pi}}\,\frac{\sin as}{s}$$

The parseval's identity is,

$$\int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|F(s)|^2ds$$

we have $\displaystyle\int_{-a}^{a}(1)^2dx = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)^2ds$

$$(x)_{-a}^{a} = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)^2ds$$

$$\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)^2ds = \frac{\pi}{2}(a+a)$$

$$2\int_{0}^{\infty}\left(\frac{\sin as}{s}\right)^2ds = \frac{\pi}{\cancel{2}}(\cancel{2}a)$$

$$\int_{0}^{\infty}\left(\frac{\sin as}{s}\right)^2ds = \frac{\pi}{2}a$$

put $as=t \Rightarrow ds = dt/a$ | when $s=0,\ t=0$; $s=\infty,\ t=\infty$.

$$\therefore \int_{0}^{\infty}\left(\frac{\sin t}{t/a}\right)^2\cdot\frac{dt}{a} = \frac{\pi}{2}a$$

$$\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^2\cancel{a^2}\cdot\frac{dt}{a} = \pi/2\, a.$$

$$\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^2dt = \pi/2$$

12b i) Find the Fourier sine transform of $f(x)=\begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2 \\ 0, & x>2. \end{cases}$ (6)

$$F_S[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 f(x) \sin sx \, dx + \int_1^2 f(x) \sin sx \, dx + \int_2^\infty f(x) \sin sx \, dx \right].$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \sin sx \, dx + \int_1^2 (2-x) \sin sx \, dx + \int_2^\infty 0 . \sin sx \, dx \right].$$

$$= \sqrt{\frac{2}{\pi}} \left[\left\{ \frac{-x\cos sx}{s} + \frac{\sin sx}{s^2} \right\}_0^1 + \left\{ -\frac{(2-x)\cos sx}{s} - \frac{\sin sx}{s^2} \right\}_1^2 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos s}{s} + \frac{\sin s}{s^2} - \frac{\sin 2s}{s^2} + \frac{\cos s}{s} + \frac{\sin s}{s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2\sin s}{s^2} - \frac{\sin 2s}{s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{s^2} \left(2\sin s - \sin 2s \right) \right] = \sqrt{\frac{2}{\pi}} \left[\frac{1}{s^2} \left(2\sin s - 2\sin s \cos s \right) \right]$$

$$F_s = \sqrt{\frac{2}{\pi}} \frac{2\sin s}{s^2} (1-\cos s)$$

12b) ii) Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ using Fourier cosine transforms of e^{-ax} and e^{-bx}.

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{s^2+a^2}$$

$$F[e^{-bx}] = \sqrt{\frac{2}{\pi}} \cdot \frac{b}{s^2+b^2}$$

We know that $\int_0^\infty F_c[f(x)] . F_c[g(x)] \, ds = \int_0^\infty f(x) . g(x) \, dx$

$$\int_0^\infty \sqrt{\frac{2}{\pi}} \cdot \frac{a}{s^2+a^2} \cdot \sqrt{\frac{2}{\pi}} \frac{b}{s^2+b^2} \, ds = \int_0^\infty e^{-ax} e^{-bx} \, dx.$$

$$\frac{2}{\pi}\int_0^\infty \frac{ab}{(s^2+a^2)(s^2+b^2)}\,ds = \int_0^\infty e^{-(a+b)x}\,dx$$

$$= \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty = 0 - \frac{1}{-(a+b)}$$

$$= \frac{1}{a+b}.$$

$$\therefore \int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}.$$

13) a) i) Form the partial differential equation by eliminating arbitrary functions f and ϕ from $z = f(x+ct) + \phi(x-ct)$.

Solution :

$$z = f(x+ct) + \phi(x-ct) \text{ — (1).}$$

$$p = \frac{\partial z}{\partial x} = f'(x+ct) + \phi'(x-ct) \text{ — (2)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct) \text{ — (3)}$$

$$q = \frac{\partial z}{\partial y} = cf'(x+ct) + (-c)\phi'(x-ct) \text{ — (4)}$$

$$t = \frac{\partial^2 z}{\partial t^2} = c^2 f''(x+ct) + c^2 \phi''(x-ct) \text{ — (5)}$$

Using (3) in (5) we get,

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2}\right)$$

This is the required pde.

13 a(ii) Solve the partial differential equation $(mz-ny)p+(nx-lz)q = ly-mx$ (7)

The given equation is of Lagrange's type $Pp+Qq=R$.

∴ The simultaneous equation is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

i.e $\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$.

Using Lagrange's multipliers l, m, n we get each ratio is equal to

$$\frac{l\,dx+m\,dy+n\,dz}{l(mz-ny)+m(nx-lz)+n(ly-mx)}$$

$l\,dx+m\,dy+n\,dz=0$ (Since the denominator is zero).

$\int l\,dx+\int m\,dy+\int n\,dz=0$

$lx+my+nz=C_1$

$\therefore u=lx+my+nz$.

Using Lagrange's multipliers x, y, z we get each of the above ratio is equal to $\frac{x\,dx+y\,dy+z\,dz}{x(mz-ny)+y(nx-lz)+z(ly-mx)}$

$x\,dx+y\,dy+z\,dz=0$. (Since the dr. is zero)

$\int x\,dx+\int y\,dy+\int z\,dz=0$

$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=0$

$x^2+y^2+z^2=C_2$

$\therefore v=x^2+y^2+z^2$.

∴ The solution of the given pde is $\phi(u,v)=0$.

$\phi(lx+my+nz,\ x^2+y^2+z^2)=0$.

13) b) i) Solve $(D^2-D'^2)z = e^{x-y}\sin(2x+3y)$.

The auxiliary eqn is $m^2-1=0 \Rightarrow m^2=+1 \Rightarrow m=\pm 1$

$m=1,-1$.

C.F is, $C.F = f_1(y+x) + f_2(y-x)$.

$$P.I = \frac{1}{D^2-D'^2} e^{x-y}\sin(2x+3y).$$

$D = D+1$ & $D' = D'-1$

$$= e^{x-y} \frac{1}{(D+1)^2-(D'-1)^2} \sin(2x+3y)$$

$$= e^{x-y} \frac{1}{(D^2+1+2D)-(D'^2+1-2D')} \sin(2x+3y)$$

$$= e^{x-y} \frac{1}{D^2+2D+2D'-D'^2} \sin(2x+3y).$$

$D^2 = -a^2 = -4$

$D'^2 = -b^2 = -9.$

$$= e^{x-y} \frac{1}{-4+2D+2D'+9} \sin(2x+3y)$$

$$= e^{x-y} \frac{1}{5+2D+2D'} \sin(2x+3y).$$

$$= e^{x-y} \frac{D}{5D+2D^2+2DD'} \sin(2x+3y) = e^{x-y} \frac{D}{5D\underbrace{-8-12}_{-20}} \sin(2x+3y)$$

$D^2=-4,\ DD'=-6$

$$= \frac{e^{x-y}}{5} \frac{D}{D-4} \sin(2x+3y) = \frac{e^{x-y}}{5} \frac{D(D+4)}{D^2-16} \sin(2x+3y)$$

$$= \frac{e^{x-y}}{5} \cdot \frac{(D^2+4D)\sin(2x+3y)}{-4-16} = \frac{e^{x-y}}{-100}\left[D^2\sin(2x+3y)+4D\sin(2x+3y)\right]$$

$$P.I = \frac{e^{x-y}}{-100}\left[-4\sin(2x+3y)+8\cos(2x+3y)\right]$$

$$\therefore z = f_1(y+x)+f_2(y-x) - \frac{e^{x-y}}{25}\left[2\cos(2x+3y) - \sin(2x+3y)\right]$$

13.b. (ii) Solve $(D^2 - 3DD' + 2D'^2 + 2D - 2D')z = x + y + \sin(2x+y)$. (8)

The given equation is $(D^2 - 3DD' + 2D'^2 + 2D - 2D')z = 0$.

i.e $(D^2 - 2DD' - DD' + D'^2 + D'^2 + 2D - 2D')z = 0$

$[D^2 - 2DD' + D'^2 - DD' + D'^2 + 2D - 2D']z = 0$

$[(D-D')^2 - D'(D-D') + 2(D-D')]z = 0$

i.e $(D-D')[D - D' - D' + 2]z = 0$

i.e $(D-D')(D - 2D' + 2)z = 0$

$\therefore C.F = e^{0x} f_1(y+x) + e^{-2x} f_2(y+2x)$.

$D(x+y) = 1$
$D^2(x+y) = 0$
$D'(x+y) = 1$
$D'^2(x+y) = 0$
$DD'(x+y) = 0$

$$P.I = \frac{1}{(D-D')(D-2D'+2)}(x+y) + \frac{1}{(D-D')(D-2D'+2)}\sin(2x+y)$$

$$= \frac{1}{D(1-D')2\left(1+\frac{D-2D'}{2}\right)}(x+y) + \frac{1}{D^2 - 3DD' + 2D'^2 + 2D - 2D'}\sin(2x+y)$$

$D^2 = -a^2 = -4$
$D'^2 = -b^2 = -1$
$DD' = -ab = -2$.

$$= \frac{1}{2D}(1-D')^{-1}\left(1+\frac{D-2D'}{2}\right)^{-1}(x+y) + \frac{1}{-4-3(-2)+2(-1)+2D-2D'}\sin(2x+y)$$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

$$= \frac{1}{2D}\left[1 + D' + D'^2 + \cdots\right]\left[1 - \left(\frac{D-2D'}{2}\right) + \left(\frac{D-2D'}{2}\right)^2 - \cdots\right](x+y) + \frac{1}{2D-2D'}\sin(2x+y)$$

$$= \frac{1}{2D}\left[1 - \frac{(D-2D')}{2} + D'\right](x+y) + \frac{1}{2}\cdot\frac{D+D'}{D^2 - D'^2}\sin(2x+y)$$

$$= \frac{1}{2D}\left[1 - \frac{D}{2} + 2D'\right](x+y) + \frac{1}{2}\cdot\frac{(D+D')}{-4-(-1)}\sin(2x+y)$$

$$= \frac{1}{2D}\left[x+y - \frac{1}{2}(1) + 2(1)\right] + \frac{1}{-6}\left[D\sin(2x+y) + D'\sin(2x+y)\right]$$

$$= \frac{1}{2D}\left(x+y+\frac{3}{2}\right) - \frac{1}{6}\left[2\cos(2x+y) + \cos(2x+y)\right]$$

$$P.I = \frac{1}{2}\left[\frac{x^2}{2} + yx + \frac{3}{2}x\right] - \frac{1}{2}\cos(2x+y)$$

Hence the complete solution is,

$$Z = C.F + P.I$$

i.e $z = f_1(y+x) + e^{-2x} f_2(y+2x) + \frac{x^2}{4} + \frac{xy}{2} + \frac{3x}{4} - \frac{\cos(2x+y)}{2}$

14) a) A uniform string is stretched and fastened to two point 'l' apart. Motion is started by displacing the string into the form of the curve $y = kx(l-x)$ and then released from this position at time $t=0$. Derive the expression for the displacement of any pt of the string at a distance x from one end at time t.

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$.

The boundary conditions are

(i) $y(0,t) = 0$ for all $t > 0$

(ii) $y(l,t) = 0$ for all $t > 0$

(iii) $\frac{\partial y}{\partial t}(x,0) = 0$, $0 < x < l$.

(iv) $y(x,0) = k(lx - x^2)$, $0 < x < l$

The correct solution which satisfies boundary conditions is,

$y(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos apt + C_4 \sin pat)$ — ①.

Applying condition (i) in ① we get

$y(0,t) = C_1 (C_3 \cos pat + C_4 \sin pat) = 0$

$C_1 = 0$ and $C_3 \cos pat + C_4 \sin pat \neq 0$.

Putting $\boxed{C_1 = 0}$ in ① we get,

$y(l,t) = C_2 \sin pl (C_3 \cos pat + C_4 \sin pat) = 0$ — ②

Here $C_3 \cos pat + C_4 \sin pat \neq 0$ $\therefore$ either $C_2 = 0$ or $\sin pl = 0$.

If $C_2 = 0$ then we get a trivial solution. $\therefore C_2 \neq 0$.

$\therefore$ we take $\sin pl = 0 \Rightarrow pl = n\pi \Rightarrow \boxed{p = n\pi/l}$

Substituting $p = \frac{n\pi}{l}$ in ② we get

$y(x,t) = C_2 \sin \frac{n\pi x}{l} \left(C_3 \cos \frac{n\pi at}{l} + C_4 \sin \frac{n\pi at}{l} \right)$ — ③

Differentiating ③ partially w.r.to 't' we get

$\frac{\partial y}{\partial t}(x,t) = C_2 \sin \frac{n\pi x}{l} \left[- C_3 \frac{n\pi a}{l} \sin \frac{n\pi at}{l} + C_4 \frac{n\pi a}{l} \cos \frac{n\pi at}{l} \right]$

Applying condition (ii) we get $\frac{\partial y(x,0)}{\partial t} = c_2 \sin\frac{n\pi x}{l}\, c_4 \frac{n\pi a}{l} = 0$.

$c_2 \neq 0$, $\sin\frac{n\pi x}{l} \neq 0$ ($\because$ It is defined for all x) and $\frac{n\pi a}{l} \neq 0$ ($\because$ all are constants)

$\boxed{\therefore c_4 = 0}$

$\therefore$ Substituting $c_4 = 0$ in ③ we get $y(x,t) = \underbrace{c_2 c_3}_{C_n} \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}$ — ④

$\therefore$ The most general solution of ④ is

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \quad \text{— ⑤}$$

($\because$ the pde is linear any linear combination of solutions of the form ④ is also a solution of the equation)

Applying (iv) in ⑤ we get

$$y(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} = k(lx - x^2), \text{ where } C_n = b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx.$$

i.e. $C_n = \frac{2}{l}\int_0^l k(\underbrace{lx - x^2}_{u}) \underbrace{\sin\frac{n\pi x}{l}}_{v}\,dx = \frac{2k}{l}\left[-(lx - x^2)\cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi} + (l-2x)\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l} - 2\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l$

$$= \frac{2k}{l}\left[-2\cdot\frac{l^3}{n^3\pi^3}\cos n\pi + \frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{4kl^3}{n^3\pi^3 l}\left[1 - (-1)^n\right] = \frac{4kl^2}{n^3\pi^2}\left[1 - (-1)^n\right] \quad \therefore C_n = \begin{cases} 0, & n = \text{even} \\ \frac{8kl^2}{n^3\pi^3}, & n = \text{odd} \end{cases}$$

Substituting ⑥ in ⑤ we get

$$y(x,t) = \sum_{n=1,3,5}^{\infty} \frac{8kl^2}{n^3\pi^3} \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \text{ //.}$$

---

14) b) A rectangular plate with insulated surface is 20 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge $x=0$ is given by $u = \begin{cases} 10y & \text{for } 0 \le y \le 10 \\ 10(20-y) & \text{for } 10 \le y \le 20 \end{cases}$ and the two long edges as well as the other short edge are kept at 0°c. Find the steady state distribution in the plate.

Two dimensional heat flow equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ — Ⓐ

Let $l = 20$ cm. The boundary conditions are

(i) $u(x,0) = 0$ for all x

(ii) $u(x,l) = 0$ for all x

(iii) $u(\infty, y) = 0$ when $x \to \infty$, $u \to 0$.

(iv) $u(0,y) = \begin{cases} \frac{ly}{2} & \text{for } 0 \le y \le l/2 \\ \frac{l}{2}(l-y) & \text{for } l/2 \le y \le l \end{cases}$

$l = 20$

$l/2 = 10$

The correct solution is

$$u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad \text{— ①}$$

Applying condition (i) in ①, we get

$$u(x,0) = (c_1 e^{px} + c_2 e^{-px}) c_3 = 0. \Rightarrow \boxed{c_3 = 0}$$

Using $c_3 = 0$ in ① we get,

$$u(x,y) = (c_1 e^{px} + c_2 e^{-px}) c_4 \sin py \quad \text{— ②}$$

Applying (ii) in ②, we get

$$u(x,l) = (c_1 e^{px} + c_2 e^{-px}) c_4 \sin pl = 0.$$

$c_1 e^{px} + c_2 e^{-px} \neq 0 \ \forall x$ & $c_4 \neq 0$ since c_3 already zero.

$\therefore \sin pl = 0 \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$

Using $p = \frac{n\pi}{l}$ in ②, we get

$$u(x,y) = \left(c_1 e^{\frac{n\pi x}{l}} + c_2 e^{-\frac{n\pi x}{l}}\right) c_4 \sin \frac{n\pi y}{l} \quad \text{— ③}$$

Applying (iii) in ③ we get,

$$u(\infty,y) = (c_1 e^{\infty} + c_2 e^{-\infty}) c_4 \sin \frac{n\pi y}{l} \Rightarrow 0 = c_1 e^{\infty} c_4 \sin \frac{n\pi y}{l}$$

$e^{-\infty} = 0$, $c_4 \neq 0$, & $\sin \frac{n\pi y}{l} \neq 0$, $e^{\infty} = \infty \neq 0$ $\therefore \boxed{c_1 = 0}$

③ $\Rightarrow u(x,y) = c_2 e^{-\frac{n\pi x}{l}} c_4 \sin \frac{n\pi y}{l} = c_2 c_4 e^{-n\pi x/l} \sin \frac{n\pi y}{l}$

$$u(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{l}} \cdot \sin \frac{n\pi y}{l} \quad \text{— ④}$$ is the most general solution.

Applying (iv) in ④, we get

$u(0,y) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi y}{l}$. This is half range sine series.

$$\therefore C_n = \frac{2}{l} \int_0^l f(y) \sin \frac{n\pi y}{l} \, dy.$$

$$= \frac{2}{l} \left[\int_0^{l/2} \frac{ly}{2} \sin \frac{n\pi y}{l} dy + \int_{l/2}^{l} \frac{l}{2}(l-y) \sin \frac{n\pi y}{l} dy \right]$$

$$= \frac{\cancel{2}}{\cancel{l}} \times \frac{\cancel{l}}{\cancel{2}} \left[\underbrace{\int_0^{l/2} y \sin \frac{n\pi y}{l} dy}_{I_1} + \underbrace{\int_{l/2}^{l} (l-y) \sin \frac{n\pi y}{l} dy}_{I_2} \right]$$

$$C_n = I_1 + I_2 = \frac{2l^2}{n^2\pi^2} \sin \frac{n\pi}{2}$$

15 b) i) Solve the difference equation $y(n+3) - 3y(n+1) + 2y(n) = 0$, given that $y(0) = 4$, $y(1) = 0$ and $y(2) = 8$.

Given $y(n+3) - 3y(n+1) + 2y(n) = 0$ — ①, $y(0) = 4$, $y(1) = 0$ and $y(2) = 8$ — ②

Taking z-transform on both sides we get $Z[y(n+3)] - 3Z[y(n+1)] + 2Z[y(n)] = 0$

$$[z^3\bar{y} - z^3y(0) - z^2y(1) - zy(2)] - 3[z\bar{y} - zy(0)] + 2\bar{y} = 0, \quad \bar{y} = Z[y(n)]$$

From ②,

$$(z^3 - 3z + 2)\bar{y} = 4z^3 - 4z \quad \Rightarrow \bar{y} = \frac{4z^3 - 4z}{z^3 - 3z + 2}$$

i.e $$\frac{\bar{y}}{z} = \frac{4z^2 - 4}{z^3 - 3z + 2} = \frac{4z^2 - 4}{(z-1)^2(z+2)}$$

$$\frac{4z^2 - 4}{(z-1)^2(z+2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z+2}$$

Put $z = 1$, $B = 0$ | $z = -2$, $C = 4/3$ | $z = 0$, $A = 8/3$

$$\therefore \frac{\bar{y}}{z} = \frac{8/3}{z-1} + \frac{4/3}{z+2}$$

$$\therefore \bar{y} = 8/3 \frac{z}{z-1} + 4/3 \frac{z}{z+2} \quad \text{i.e } Z[y(n)] = \frac{8}{3}\frac{z}{z-1} + \frac{4}{3}\cdot\frac{z}{z+2}$$

$$\therefore y(n) = 8/3\, z^{-1}\left[\frac{z}{z-1}\right] + 4/3\, z^{-1}\left[\frac{z}{z+2}\right]$$

i.e $y(n) = 8/3\,(1)^n + \frac{4}{3}(-2)^n, \quad n \geq 0$

ii) Derive the difference equation from $y_n = (A + Bn)(-3)^n$.

Given $y_n = A(-3)^n + Bn(-3)^n \Rightarrow y_{n+1} = A(-3)^{n+1} + B(n+1)(-3)^{n+1}$

$= -3A(-3)^n - 3B(n+1)(-3)^{n+1}$

and $y_{n+2} = A(-3)^{n+2} + B(n+2)(-3)^{n+2}$

$= 9A(-3)^n + 9B(n+2)(-3)^n$

From y_n, y_{n+1} & y_{n+2}, eliminating A & B we get

$$\begin{vmatrix} y_n & 1 & n \\ y_{n+1} & -3 & -3(n+1) \\ y_{n+2} & 9 & 9(n+2) \end{vmatrix} = 0$$

i.e $y_n[-27(n+2) + 27(n+1)] - 1[9y_n(n+2) + 3(n+1)y_{n+2}] + n[9y_{n+1} + 3y_{n+2}] = 0$

$$3y_{n+2} + 9n\,y_{n+1} - (35 + 9n)y_n = 0$$

$$I_1 = \int_0^{l/2} y \sin\frac{n\pi y}{l}\,dy = \left[-y\cos\frac{n\pi y}{l}\cdot\frac{l}{n\pi} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi y}{l}\right]_0^{l/2} = \left[\left\{-\frac{l^2}{n\pi}\cos n\pi/2 + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\} - \{0+0\}\right]$$

$$I_1 = \frac{-l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$I_2 = \int_{l/2}^{l} (l-y)\sin\frac{n\pi y}{l}\,dy = \left[-(l-y)\frac{l}{n\pi}\cos\frac{n\pi y}{l} + (-1)\frac{l^2}{n^2\pi^2}\sin\frac{n\pi y}{l}\right]_{l/2}^{l}$$

$$= \left[\left\{0 - \cancelto{0}{\frac{l^2}{n^2\pi^2}\sin n\pi}\right\} - \left\{-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}\right]$$

$$\therefore I_2 = \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin n\pi/2.$$

$$\therefore C_n = \frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

$$\therefore u(x,y) = \sum_{n=1}^{\infty} \frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\, e^{-n\pi x/l}\sin\frac{n\pi y}{l}$$

15) b) (i) Using convolution theorem, find inverse z-transform of $\frac{z^2}{(z-1)(z-3)}$.

$$z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = z^{-1}\left[\frac{z}{z-1}\cdot\frac{z}{z-3}\right]$$

$$= 1^n * 3^n \qquad \text{since } z(1^n) = \frac{z}{z-1}, \quad z[3^n] = \frac{z}{z-3}$$

$$= \sum_{r=0}^{n} (1)^r 3^{n-r}$$

$$= 3^n + 3^{n-1} + 3^{n-2} + \cdots + 3^1 + 1$$

$$= 1 + 3 + 3^2 + \cdots + 3^n$$

$$= \frac{3^{n+1}-1}{3-1} = \frac{3^{n+1}-1}{2}$$

$$\therefore z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = \frac{3^{n+1}-1}{2}.$$

(ii) Find the z-transform of $a^n\cos n\theta$ and $e^{-at}\sin bt$

$$z[a^n f(n)] = F(z/a), \quad z[f(n)] = F(z).$$

$$\therefore z[a^n\cos n\theta] = z[\cos n\theta]_{z\to z/a} = \left[\frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}\right]_{z\to z/a}$$

$$= \frac{z/a\,(z/a - \cos\theta)}{z^2/a^2 - 2z/a\,\cos\theta + 1}$$

$$\therefore z[a^n\cos n\theta] = \frac{z(z-a\cos\theta)}{z^2 - 2az\cos\theta + a^2}.$$